Problems 2 and 5 are for homework. Solutions must be submitted before the next lecture (not tutorial!) to be evaluated. Students are not allowed to keep submitted solutions after evaluation.

Problem 1. Find the dual problem to the following linear programming problems and write the complementary slackness conditions.

1. $\max c^T x$ subject to $Ax \le b$,
2. $\max c^T x$ subject to $Ax = b, x \ge 0$,
3. $\min c^T x$ subject to $A_1 x = b_1, A_2 x \ge b_2$.

Problem 2. Find dual problems to the following two linear programming problems (which have the same constrains) and write the complementary slackness conditions.

$$\begin{array}{rrcrcrcrcl}
\text{1) Maximize} & x_1 & - & 2x_2 & & & + & 3x_4 & & \\
\text{2) Minimize} & x_1 & - & 2x_2 & & & + & 3x_4 & & \\
\text{subject to} & & & x_2 & - & 6x_3 & + & x_4 & \le & 4 \\
 & -x_1 & + & 3x_2 & - & 3x_3 & & & = & 0 \\
 & 6x_1 & - & 2x_2 & + & 2x_3 & - & 4x_4 & \ge & 5 \\
 & \multicolumn{9}{l}{x_2 \le 0,\ x_4 \ge 0}
\end{array}$$

Problem 3. Write an integer linear programming problem for minimal weight perfect matching problem of a graph. Consider the relaxation of that matching problem (i.e. ignore all integral conditions). Find the dual problem of this relaxed problem and write the complementary slackness conditions (for the relaxed and the dual problem).

Problem 4. The problem is to find the distance between two polyhedrons

$$P^1 = \left\{\boldsymbol{x}^1 \in \mathbb{R}^n;\ A^1\boldsymbol{x}^1 \le \boldsymbol{b}^1\right\} \text{ and } P^2 = \left\{\boldsymbol{x}^2 \in \mathbb{R}^n;\ A^2\boldsymbol{x}^2 \le \boldsymbol{b}^2\right\}$$

in the Postman (L_1) metric. The distance of two sets $P_1, P_2 \in \mathbb{R}^n$ is the distance of two closest points $\boldsymbol{x}^1 \in P^1$ and $\boldsymbol{x}^2 \in P^2$. The distance of two points $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in the L_1 metric is $\sum_{i=1}^n |\boldsymbol{x}^1_i - \boldsymbol{x}^2_i|$.

Formulate this problem using linear programming. Then, write the dual problem and complementary slackness conditions.

Problem 5 (The shortest path in a graph)**.** Given a weighted graph (V, E, f) where $f : E \to \mathbb{R}^+$, formulate the problem of finding the lengths of shortest paths from a given starting vertex to all other ones using linear programming. Then, write the dual problem and complementary slackness conditions.

Problem 6. Prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}} A = 0$ and $\boldsymbol{y}^{\mathrm{T}} b = -1$ has no solution.

Problem 7. Let $A \in R^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Prove the following statements.

1. The system $A\boldsymbol{x} \le \boldsymbol{b}$ is infeasible if and only if $\mathbf{0}\boldsymbol{x} \le -1$ is a non-negative linear combination of inequalities $A\boldsymbol{x} \le \boldsymbol{b}$.
2. The system $A\boldsymbol{x} \le \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every non-negative $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}} A \ge \mathbf{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \ge 0$.
3. The system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}} A \ge \mathbf{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \ge 0$.