Problem 1 is for homework. Solutions must be submitted before the next lecture (not tutorial!) to be evaluated. Students are not allowed to keep submitted solutions after evaluation.

Problem 1. Let M be a matching of G and let p be the cardinality of the maximum matching. Prove that there are at least $p - |M|$ vertex-disjoint M-augmenting paths.

Problem 2. Let (V, E, ω) be an edge-weighted graph and let $\omega'(e) = \omega(e) + r$ for every $e \in E$ where r is a real number. Prove that every minimal-weight perfect matching of the graph (V, E, ω) is a minimal-weight perfect matching of the graph (V, E, ω'). Does this statement also hold for maximal-weight (general) matchings?

Problem 3. For every $n \geq 3$ find a connected graph on n vertices such that the relaxed linear programming problem for perfect matching ($\{\boldsymbol{x} \in \mathbb{R}^E;\ A\boldsymbol{x} = 1,\ \boldsymbol{x} \geq \mathbf{0}\}$ where A is the incidence matrix) has no feasible solution.

Problem 4. Let $G = (V, E)$ be a graph with weights $c \in \mathbb{R}^E$ and let k be an integer. A k-matching in G is a matching of cardinality k. Using the algorithm for minimum-weight perfect matching find minimum-weight k-matching.

Problem 5. Let M be a perfect matching of $G = (V, E)$ with weights $c \in \mathbb{R}^E$. An even cycle C of G is M-alternating if its edges are alternately in and not in M. The cost of M-alternating cycle C is $\sum_{e \in C \setminus M} c_e - \sum_{e \in C \cap M} c_e$. Prove that M is of minumum weight with respect to c if and only if there is no M-alternating cycle of negative cost.

Problem 6. Slither is a two-person game played on a graph $G = (V, E)$. The players play alternatively. At each step the player whose turn it is chooses a previously unchosen edge. The only rule is that at every step the set of chosen edges forms a path. The loser is the player unable to extern the path. Prove that, if G has a perfect matching, then the first player has a winning strategy.

Problem 7. Prove that the linear programming

$$\begin{array}{lrcll} \text{Minimize} & \boldsymbol{cx} & & & \\ \text{subject to} & \delta^u \boldsymbol{x} & = & 1 & \text{for all } u \in V \\ & \delta^D \boldsymbol{x} & \geq & 1 & \text{for all } D \in \mathcal{C} \\ & \boldsymbol{x} & \geq & \mathbf{0} & \end{array}$$

is feasible if and only if G has a perfect matching (without using algorithms from the lecture). Also prove the convex hull of characteristic vectors of perfect matchings is exactly the set of all feasible solution this set of linear inequalities.

Problem 8. An edge cover of a graph $G = (V, E)$ without isolated vertices is a set of edges D such that vertex of G is incident with at least one edge of D. Prove that is size of maximum matching plus the size of minimum edge cover equals to the number of vertices. Find an algorithm for the minimum-weight edge cover problem.

Problem 9. Consider a graph $G = (V, E)$ and the corresponding relax linear programming problem of perfect matching.

$$\min \sum_{e \in E} c_e \cdot x_e$$

$$\sum_{u \in V : uv \in E} x_{uv} = 1,\ \forall v \in V \tag{1}$$

$$x_{uv} \geq 0,\ \forall uv \in E$$

1. For every $n \geq 3$ find a connected graph on n vertices such that (1) has no feasible solution.
2. For every $n \geq 3$ find a connected graph on n vertices such that (1) has a feasible solution.
3. Prove that if there exists $E' \subseteq E$ such that every component of (V, E') is an odd cycle or an isolated edge, then (1) has a feasible solution.
4. A vector $\boldsymbol{x}$ is called half-intergral if $2\boldsymbol{x}$ is an integral vector. Prove that if (1) has an half-integral feasible solution, then there exists $E' \subseteq E$ such that every component of (V, E') is an odd cycle or an isolated edge.
5. Prove that if (1) has a feasible solution, then there exists a half-integral feasible solution.