The first homework is Problem 7.

Problem 1 (Bakery)**.** A bakery produces four products: bread, bagels, baguettes and donuts. To bake a single bread, they need 500g of flour, 10 eggs and 50 grams of salt. To bake a bagel, they need 150 grams of flour, 2 eggs and 10g of salt. For a baguette, they need 230g of flour, 7 eggs and 15g of salt. For a donut, they need 100g of flour and 1 egg. The bakery has a daily supply of 5 kg of flour, 125 eggs, and 500g of salt.

The bakery charges 20 CZK for one bread, 2 CZK for a bagel, 10 CZK for a baguette and 7 CZK for a donut. The bakery tries to maximize its profit. Formulate a linear program that suggests the right amount of bread, bagels, baguettes and donuts that the bakery should produce. (You don't have to solve it!)

Problem 2. Find all solutions of the following system of linear equations.

$$\begin{aligned} -x_1+2x_2+\ x_3+4x_4 &= 3 \\ -2x_1+4x_2+\ x_3+7x_4 &= 5 \\ x_1-2x_2+\ x_3-2x_4 &= -1 \\ -x_1+2x_2+2x_3+5x_4 &= 4 \end{aligned}$$

Problem 3. For a non-empty set $M \subseteq \mathbb{R}^n$, the following conditions are equivalent.

1. M is a linear (= vector) space.
2. $a\boldsymbol{x} + b\boldsymbol{y} \in M$ for every $a, b \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in M$
3. M is the set of all linear combinations of some set $S \subseteq \mathbb{R}^n$.
4. M is the set of all solutions of some consistent homogeneous system of linear equations ($A\boldsymbol{x} = \mathbf{0}$).

Problem 4. Prove that $A + \boldsymbol{x}$ is an affine space for every affine space $A \subseteq \mathbb{R}^n$ and every vector $\boldsymbol{x} \in \mathbb{R}^n$.

Problem 5. Prove that if $A \subseteq \mathbb{R}^n$ is an affine space, then $A - \boldsymbol{x}$ is a linear space for for every $\boldsymbol{x} \in A$. Furthermore, all spaces $A - \boldsymbol{x}$ are the same for all $\boldsymbol{x} \in A$.

Problem 6. Prove that an affine space $A \subseteq \mathbb{R}^n$ is linear if and only if A contains the origin $\mathbf{0}$.

Problem 7. Prove that a set $A \subseteq \mathbb{R}^n$ is affine if and only if for every pair of points of A the line defined by those two points is contained in A.

Problem 8. Prove that the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ is an affine space and every affine space is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ for some A and $\boldsymbol{b}$, assuming $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 9. Prove that the following statements holds.

- The intersection of linear spaces is also a linear space.
- The intersection of affine spaces is an affine space or empty.
- The intersection of convex sets is also a convex set.

Problem 10. Prove that for vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ the following statements are equivalent.

- Vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affinely independent.
- Vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0$ are linearly independent.
- The origin $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \boldsymbol{v}_i$ such that $\sum \alpha_i = 0$ and $\boldsymbol{\alpha} \neq \mathbf{0}$.

Problem 11. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 12. Prove that all affine bases of an affine space have the same cardinality.

Problem 13. Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S.