The second homework is Problem 3 and Problem 7.

Problem 1 (Independent set)**.** A set of vertices S of a graph (V, E) is called *independent* if no two vertices of S are joined by an edge. Formulate the problem of finding the maximal independent set using Integer Linear Programming.

Problem 2 (Matching)**.** A set of edges M of a graph (V, E) is called *matching* if every vertex is covered by at most one edge of M. A matching is *perfect* if every vertex is covered by exactly one edge of M. A weight of matching M is the sum of weights $w(e)$ over all edge $e \in M$. Using Integer Linear Programming formulate the following two problems.

1. Find the minimal-weight perfect matching.
2. Find the maximal-weight matching.

Problem 3. Show stat satisfiability of a given boolean formula in Conjunctive Normal Form can be solved using Integer Linear Programming.

Problem 4 (Connectivity)**.** Formulate using Linear Programming the decision problem whether a given graph is connected.

Problem 5 (The Classic Transportation Problem I)**.** An unknown country has n bakeries and m shops. Every day, the i-th bakery bakes b_i breads and the j-the shop sells s_j breads. The transportation of one bread from the i-th bakery to the j-th shop cost $c_{i,j}$. Describe the following problem using linear programming: Find the cheapest transportation of all breads so that all breads are transported according to the number of baked and sold breads.

Furthermore, find necessary and sufficient conditions for the existence of a feasible and an optimal solutions.

Problem 6 (The Classic Transportation Problem II)**.** It turned out that whenever i-th bakery supplies j-th shop (with positive number of breads), logistic is needed to organize the transportation which cost an extra $l_{i,j}$. Using linear programming formulate the problem of minimizing the total transportation and logistic expenses.

Problem 7. The problem is to find the distance between two polyhedrons

$$P^1 = \left\{\boldsymbol{x}^1 \in \mathbb{R}^n;\ A^1\boldsymbol{x}^1 \le \boldsymbol{b}^1\right\} \text{ and } P^2 = \left\{\boldsymbol{x}^2 \in \mathbb{R}^n;\ A^2\boldsymbol{x}^2 \le \boldsymbol{b}^2\right\}$$

in the Postman (L_1) metric. The distance of two sets $P_1, P_2 \in \mathbb{R}^n$ is the distance of two closest points $\boldsymbol{x}^1 \in P^1$ and $\boldsymbol{x}^2 \in P^2$. The distance of two points $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in the L_1 metric is $\sum_{i=1}^n |\boldsymbol{x}_i^1 - \boldsymbol{x}_i^2|$. Write this problem both in the canonical and the equation forms.

Problem 8 (Sudoku)**.** Sudoku can be easily solved using a backtrack. Is it also possible to solve it using Linear programming?

Problem 9. Plan a production of chocolate for the next year so that the total cost is minimal. The predicted demand of chocolate during the i-th month is d_i units. The change of the production between two consecutive month cost 1500 CZK per unit. Storing chocolate from one month to the following one cost 600 CZK per unit. Chocolate can be stored at most one month because shelf life. As usually, formulate this problem using linear programming.

Is it necessary to consider the production cost?

Problem 10. Write the following problems both in the canonical and the equation forms.

$$\begin{array}{lrcrcl} \text{Maximize} & 2\boldsymbol{x}_1 & - & 3\boldsymbol{x}_2 & & \\ \text{subject to} & 4\boldsymbol{x}_1 & - & 5\boldsymbol{x}_2 & \le & 6 \\ & 7\boldsymbol{x}_1 & + & 8\boldsymbol{x}_2 & = & 8 \\ & \boldsymbol{x}_1 \ge 0. & & & & \end{array}$$

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A'\boldsymbol{x} \ge \boldsymbol{b}' \\ & A''\boldsymbol{x} = \boldsymbol{b}'' \\ & \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \ge 0 \end{array}$$

where$A' \in \mathbb{R}^{m' \times n}$, $A'' \in \mathbb{R}^{m'' \times n}$, $\boldsymbol{b}' \in \mathbb{R}^{m'}$, $\boldsymbol{b}'' \in \mathbb{R}^{m''}$, $\boldsymbol{c} \in \mathbb{R}^n$.

Problem 11. Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr} 5x & + & 3y & & & & & \leq & 8 \\ 2x & & & - & 5z & & & < & -3 \\ 6x & + & 5y & & & + & 2w & = & 5 \\ & & & & 3z & + & 2w & > & 5 \\ & & & & & & x,y,z,w & \geq & 0 \end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

Problem 12. Using the graphical methods find the optimal solutions of two objective functions

- min $\boldsymbol{x}_1 + \boldsymbol{x}_2$
- max $\boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following conditions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$