The third homework is Problem 1.

Problem 1. Prove that for vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ the following statements are equivalent.

- Vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affinely independent.
- Vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0$ are linearly independent.
- The origin $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \boldsymbol{v}_i$ such that $\sum \alpha_i = 0$ and $\boldsymbol{\alpha} \neq \mathbf{0}$.

Problem 2. The set of all affine combinations is an affine space.

Problem 3. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 4. Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S.

Problem 5. Let $S \subseteq \mathbb{R}^n$. Prove that $\operatorname{span}(S) = \operatorname{aff}(S \cup \{\mathbf{0}\})$.

Problem 6 (Sudoku)**.** Sudoku can be easily solved using a backtrack. Is it also possible to solve it using Linear programming?

Problem 7. Plan a production of chocolate for the next year so that the total cost is minimal. The predicted demand of chocolate during the i-th month is d_i units. The change of the production between two consecutive month cost 1500 CZK per unit. Storing chocolate from one month to the following one cost 600 CZK per unit. Chocolate can be stored at most one month because shelf life. As usually, formulate this problem using linear programming.

Is it necessary to consider the production cost?

Problem 8. Write the following problems both in the canonical and the equation forms.

$$\begin{array}{lrcrcl}
\text{Maximize} & 2\boldsymbol{x}_1 & - & 3\boldsymbol{x}_2 & & \\
\text{subject to} & 4\boldsymbol{x}_1 & - & 5\boldsymbol{x}_2 & \leq & 6 \\
& 7\boldsymbol{x}_1 & + & 8\boldsymbol{x}_2 & = & 8 \\
& \multicolumn{5}{l}{\boldsymbol{x}_1 \geq 0.}
\end{array}$$

$$\begin{array}{ll}
\text{Maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\
\text{subject to} & A'\boldsymbol{x} \geq \boldsymbol{b}' \\
& A''\boldsymbol{x} = \boldsymbol{b}'' \\
& \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \geq 0
\end{array}$$

where$A' \in \mathbb{R}^{m' \times n}$, $A'' \in \mathbb{R}^{m'' \times n}$, $\boldsymbol{b}' \in \mathbb{R}^{m'}$, $\boldsymbol{b}'' \in \mathbb{R}^{m''}$, $\boldsymbol{c} \in \mathbb{R}^n$.

Problem 9. Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr}
5x & + & 3y & & & & & \leq & 8 \\
2x & & & - & 5z & & & < & -3 \\
6x & + & 5y & & & + & 2w & = & 5 \\
& & & & 3z & + & 2w & > & 5 \\
& & & & \multicolumn{3}{r}{x, y, z, w} & \geq & 0
\end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

Problem 10. Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$

- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following conditions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$