**The fourth homework is Problem 3.**

**Problem 1.** A square at the center in the point $(r_1, r_2)$ with sides of length $s$ is the square with vertices $\{(r_1 + s_1, r_2 + s_2);\ s_1, s_2, \in \{-s/2, s/2\}\}$.

An input of the problem is a set of points $b_1, \ldots, b_n$. For every point $b_i$, we have to assign a square at the center $b_i$ with sides of length $a_i >= 0$. The area of the intersection of every pair of squares has to be zero, i.e. the intersection has to be at most an side. For instance, you can assign to points $(0, 0)$ and $(5, 4)$ squares with sides of length 2 and 1, or 6 and 4, or 10 and 0; however, sides of length 4 and 7 are forbidden. If a square has a side of length 0, then it is allowed to lie on a side (vertex) of a square but it must not lie inside another square.

Model this problem using linear programming.

**Problem 2.** The travelling salesperson problem (TSP) asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Formulate the Travelling salesman problem using (integer) linear programming.

**Problem 3.** A steel factory produces wire ropes of length $L$ which is bought by an electricity distributor to reinforcement various parts of the grid. The distributor estimates that cables of length $a_1, \ldots, a_n$ are needed for the reinforcement (assume that $a_1, \ldots, a_n \leq L$). So, the question is how to cut wire ropes of length $L$ into cables of length $a_1, \ldots, a_n$ so that the sum of length of unused parts is minimal. Model this problem using a single Integer linear programming problem.

**Problem 4.** Prove that vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent if and only if vectors $\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are affinely independent.

**Problem 5.** Let $S$ be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, $B$ is a linear base of $S$ if and only if $B \cup \{\mathbf{0}\}$ is an affine base of $S$.

**Problem 6.** Let $S \subseteq \mathbb{R}^n$. Prove that $\operatorname{span}(S) = \operatorname{aff}(S \cup \{\mathbf{0}\})$.

**Problem 7.** Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr}
5x & + & 3y & & & & & \leq & 8 \\
2x & & & - & 5z & & & < & -3 \\
6x & + & 5y & & & + & 2w & = & 5 \\
& & & & 3z & + & 2w & > & 5 \\
& & & & & x, y, z, & w & \geq & 0
\end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

**Problem 8.** Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$
- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following conditions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$