The fifth homework is Problem 5.

Problem 1. Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr} 5x & + & 3y & & & & & \leq & 8 \\ 2x & & & - & 5z & & & < & -3 \\ 6x & + & 5y & & & + & 2w & = & 5 \\ & & & & 3z & + & 2w & > & 5 \\ & & & & & & x,y,z,w & \geq & 0 \end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

Problem 2. Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$
- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following conditions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$

Definition 1. Let P be a polyhedron. A half-space $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \leq \beta$ is called a *supporting hyperplane* of P if the inequality $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \leq \beta$ holds for every $x \in P$ and the hyperplane $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} = \beta$ has a non-empty intersection with P.

Definition 2. If $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \leq \beta$ is a supporting hyperplane of a polyhedron P, then $P \cap \left\{\boldsymbol{x};\ \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} = \beta\right\}$ is called a *face* of P.

By convention, the empty set and P are also called faces, and the other faces are *proper* faces.

Definition 3. Let P be a d-dimensional polyhedron.

- A 0-dimensional face of P is called a *vertex* of P.
- A 1-dimensional face is of P called an *edge* of P.
- A $(d-1)$-dimensional face of P is called an *facet* of P.

Problem 3. The intersection of two faces of a polyhedron P is a face of P.

Problem 4. Let n-dimensional hypercube be the set $\{\boldsymbol{x} \in \mathbb{R}^n;\ \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$. Let n-dimensional simplex be the convex hull of $n+1$ affinely independent points such that no point belongs into the convex hull of the other points. Determine the number of k-dimensional faces of the n-dimensional hypercube and the n-dimensional simplex.

Problem 5. First, prove that the following two definitions of the n-dimensional crosspolytope are equivalent.

- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \sum_{i=1}^n |\boldsymbol{x}_i| \leq 1\}$
- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{d}\boldsymbol{x} \leq 1$ for all $\boldsymbol{d} \in \{-1,1\}^n\}$

Second, prove that the number of k-dimensional faces of the crosspolytope is $2^{k+1}\binom{n}{k+1}$.

Lemma 1. If the objective function of a linear program in the equation form is bounded above, then for every feasible solution $\boldsymbol{x}'$ there exists a basis feasible solution $\boldsymbol{x}^\star$ with the same or larger value of the objective function, i.e. $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^\star \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}'$.

In order to prove the lemma, consider $\boldsymbol{x}^\star$ be a feasible solution with $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^\star \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}'$ (e.g. $\boldsymbol{x}^\star$ for beginning). Let $K = \left\{j \in \{1,\dots,n\};\ \boldsymbol{x}^\star_j > 0\right\}$ and let $N = \{1,\dots,n\} \setminus K$. Let P be be the set of all feasible solutions satisfying $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. Prove the following steps.

1. If A_K has linearly independent columns, then $\boldsymbol{x}^\star$ is a basis solution and the lemma is follows.
2. There exists a non-zero vector $\boldsymbol{v}_K$ such that $A_K\boldsymbol{v}_K = \boldsymbol{0}$. Let $\boldsymbol{v}_N = \boldsymbol{0}$.
3. Explain why we can assume that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{v} \geq \boldsymbol{0}$.
4. Consider the line $x(t) = \boldsymbol{x}^\star + t\boldsymbol{v}$ for $t \in \mathbb{R}$. Prove that for every $t \in \mathbb{R}$ it holds that $Ax(t) = \boldsymbol{b}$ and $(x(t))_N = \boldsymbol{0}$.
5. Prove that $\boldsymbol{c}^{\mathrm{T}}x(t) \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ holds for every $t \geq 0$.
6. Prove that the intersection of P and the line $x(t)$ is a line, line segment or a half line segment. How we can distinguish these cases and find both end-points if exists.