**The ninth homework are Problems 1 and 2.**

**Problem 1.** The problem is to find the distance between two polyhedrons

$$P^1 = \left\{\boldsymbol{x}^1 \in \mathbb{R}^n;\ A^1\boldsymbol{x}^1 \leq \boldsymbol{b}^1\right\} \text{ and } P^2 = \left\{\boldsymbol{x}^2 \in \mathbb{R}^n;\ A^2\boldsymbol{x}^2 \leq \boldsymbol{b}^2\right\}$$

in the Postman ($L_1$) metric. The distance of two sets $P_1, P_2 \in \mathbb{R}^n$ is the distance of two closest points $\boldsymbol{x}^1 \in P^1$ and $\boldsymbol{x}^2 \in P^2$. The distance of two points $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in the $L_1$ metric is $\sum_{i=1}^n |\boldsymbol{x}_i^1 - \boldsymbol{x}_i^2|$.

Formulate this problem using linear programming. Then, write the dual problem and complementary slackness conditions.

**Problem 2** (The shortest path in a graph)**.** Given a weighted graph $(V, E, f)$ where $f : E \to \mathbb{R}^+$, formulate the problem of finding the lengths of shortest paths from a given starting vertex to all other ones using linear programming. Then, write the dual problem and complementary slackness conditions.

**Problem 3.** Solve the following problem using duality and Fourier-Motzkin elimination

$$\begin{array}{lrcrcl} \text{Maximize} & & & x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 0 \\ & x_1 & & & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

**Problem 4.** Let $A \in R^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Prove the following statements.

1. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ is infeasible if and only if $\boldsymbol{0}\boldsymbol{x} \leq -1$ is a non-negative linear combination of inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$.
2. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every non-negative $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}}A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.
3. The system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}}A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.

**Problem 5.** Prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}}A = 0$ and $\boldsymbol{y}^{\mathrm{T}}b = -1$ has no solution.