The tenth homework is Problem 8.

Problem 1. Let $A \in R^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Prove the following statements.

1. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ is infeasible if and only if $\boldsymbol{0x} \leq -1$ is a non-negative linear combination of inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$.
2. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every non-negative $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}} A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.
3. The system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}} A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.

Problem 2. Let (V, E, ω) be an edge-weighted graph and let $\omega'(e) = \omega(e) + r$ for every $e \in E$ where r is a real number. Prove that every minimal-weight perfect matching of the graph (V, E, ω) is a minimal-weight perfect matching of the graph (V, E, ω'). Does this statement also hold for maximal-weight (general) matchings?

Problem 3. Let $G = (V, E)$ be a graph with weights $c \in \mathbb{R}^E$ and let k be an integer. A k-matching in G is a matching of cardinality k. Using the algorithm for minimum-weight perfect matching find minimum-weight k-matching.

Problem 4. Prove that A is totally unimodular if and only if $(A|I)$ is unimodular.

Problem 5. Prove that the incidence matrix A of a graph G is totally unimodular if and only if G is bipartite.

Problem 6. Prove that the following statements about a matrix A are equivalent.

1. A is totally unimodular.
2. A^{T} is totally unimodular.
3. $(A|I)$ is totally unimodular.
4. $(A|A)$ is totally unimodular.
5. $(A| - A)$ is totally unimodular.

Problem 7. Find a 0–1 matrix A and an integral vector $\boldsymbol{b}$ such that $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is an integral polytope but A is not totally unimodular.

Problem 8. Let n be an integer and let S be the set of all non-negative $n \times n$ matrices such that the sum of elements in every row and column is one. Prove that S is an integral polytope. Which matrices are vertices of S? For formal purposes, consider an $n \times n$ matrix as a vector in $\mathbb{R}^{n^2}$.

Problem 9. Let A be totally unimodular $m \times n$ matrix with full row rank. Let B be a basis of A (that is, a regular $n \times n$ submatrix of A). Prove that $B^{-1}A$ is totally unimodular.

Problem 10. Prove that if a matrix A and a vector $\boldsymbol{b}$ are rational, then every vertex of the polyhedron $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}\}$ is rational. Next, prove that if every vertex of a polytope P is rational, then there exists rational A and $\boldsymbol{b}$ such that $P = \{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}\}$.