1 Farkas lemma

Problem 1. Prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}}A = 0$ and $\boldsymbol{y}^{\mathrm{T}}b = -1$ has no solution.

Problem 2. Let $A \in R^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Prove the following statements.

1. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ is infeasible if and only if $\boldsymbol{0x} \leq -1$ is a non-negative linear combination of inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$.
2. The system $A\boldsymbol{x} \leq \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every non-negative $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}}A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.
3. The system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}}A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$.

2 Matchings in graphs

Problem 3. Let M be a perfect matching of $G = (V, E)$ with weights $c \in \mathbb{R}^E$. An even cycle C of G is M-alternating if its edges are alternately in and not in M. The cost of M-alternating cycle C is $\sum_{e\in C\setminus M} c_e - \sum_{e\in C\cap M} c_e$. Prove that M is of minumum weight with respect to c if and only if there is no M-alternating cycle of negative cost.

Problem 4. Let M be a matching of G and let p be the cardinality of the maximum matching. Prove that there are at least $p - |M|$ vertex-disjoint M-augmenting paths.

Problem 5. Consider the linear programming for Minimum-Weight perfect matchings in general graphs:

$$\begin{array}{lrcll} \text{Minimize} & \boldsymbol{cx} & & & \\ \text{subject to} & \delta^u\boldsymbol{x} & = & 1 & \text{for all } u \in V \\ & \delta^D\boldsymbol{x} & \geq & 1 & \text{for all } D \in \mathcal{C} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} & \end{array}$$

Where $\delta^D \in \{0,1\}^E$ is a vector such that $\delta^D_{uv} = 1$ if $|uv \cap D| = 1$ and $\delta^w = \delta^{\{w\}}$ and $\mathcal{C}$ is the set of all odd-size subsets of V.

From conditions $\delta^u\boldsymbol{x} = 1$ and $\boldsymbol{x} \geq \boldsymbol{0}$ derive using Gomory-Chvátal cutting planes inequalities $\delta^D\boldsymbol{x} \geq 1$.

Problem 6. Slither is a two-person game played on a graph $G = (V, E)$. The players play alternatively. At each step the player whose turn it is chooses a previously unchosen edge. The only rule is that at every step the set of chosen edges forms a path. The loser is the player unable to extern the path. Prove that, if G has a perfect matching, then the first player has a winning strategy.

Problem 7. Prove that the linear programming

$$\begin{array}{lrcll} \text{Minimize} & \boldsymbol{cx} & & & \\ \text{subject to} & \delta^u\boldsymbol{x} & = & 1 & \text{for all } u \in V \\ & \delta^D\boldsymbol{x} & \geq & 1 & \text{for all } D \in \mathcal{C} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} & \end{array}$$

is feasible if and only if G has a perfect matching (without using algorithms from the lecture). Also prove the convex hull of characteristic vectors of perfect matchings is exactly the set of all feasible solution this set of linear inequalities.

Problem 8. For every $n \geq 3$ find a connected graph on n vertices such that the relaxed linear programming problem for perfect matching ($\{\boldsymbol{x} \in \mathbb{R}^E;\ A\boldsymbol{x} = 1, \boldsymbol{x} \geq \boldsymbol{0}\}$ where A is the incidence matrix) has no feasible solution.

Problem 9. Let $G = (V, E)$ be a graph with weights $c \in \mathbb{R}^E$ and let k be an integer. A k-matching in G is a matching of cardinality k. Using the algorithm for minimum-weight perfect matching find minimum-weight k-matching.

Problem 10. An edge cover of a graph $G = (V, E)$ without isolated vertices is a set of edges D such that vertex of G is incident with at least one edge of D. Prove that is size of maximum matching plus the size of minimum edge cover equals to the number of vertices. Find an algorithm for the minimum-weight edge cover problem.

Problem 11. Let (V, E, ω) be an edge-weighted graph and let $\omega'(e) = \omega(e) + r$ for every $e \in E$ where r is a real number. Prove that every minimal-weight perfect matching of the graph (V, E, ω) is a minimal-weight perfect matching of the graph (V, E, ω'). Does this statement also hold for maximal-weight (general) matchings?

Problem 12. Consider a graph $G = (V, E)$ and the corresponding relax linear programming problem of perfect matching.

$$\begin{aligned} \min \sum_{e \in E} c_e \cdot x_e \\ \sum_{u \in V : uv \in E} x_{uv} = 1, \ \forall v \in V \\ x_{uv} \geq 0, \ \forall uv \in E \end{aligned} \tag{1}$$

1. For every $n \geq 3$ find a connected graph on n vertices such that (1) has no feasible solution.
2. For every $n \geq 3$ find a connected graph on n vertices such that (1) has a feasible solution.
3. Prove that if there exists $E' \subseteq E$ such that every component of (V, E') is an odd cycle or an isolated edge, then (1) has a feasible solution.
4. A vector $\boldsymbol{x}$ is called half-intergral if $2\boldsymbol{x}$ is an integral vector. Prove that if (1) has an half-integral feasible solution, then there exists $E' \subseteq E$ such that every component of (V, E') is an odd cycle or an isolated edge.
5. Prove that if (1) has a feasible solution, then there exists a half-integral feasible solution.

3 Unimodularity

Problem 13. Consider a 0–1 matrix A in which for every row, the 1s appear consecutively (that is, for every i there exists j_1, j_2 such that for every j it holds that $A_{i,j} = 1$ if and only if $j_1 \leq j \leq j_2$). Prove that the matrix A is totally unimodular.

Problem 14. Prove that A is totally unimodular if and only if $(A|I)$ is unimodular.

Problem 15. Prove that the incidence matrix A of a graph G is totally unimodular if and only if G is bipartite.

Problem 16. Find a 0–1 matrix A and an integral vector $\boldsymbol{b}$ such that $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is an integral polytope but A is not totally unimodular.

Problem 17. Let n be an integer and let S be the set of all non-negative $n \times n$ matrices such that the sum of elements in every row and column is one. Prove that S is an integral polytope. Which matrices are vertices of S? For formal purposes, consider an $n \times n$ matrix as a vector in $\mathbb{R}^{n^2}$.

Problem 18. Let A be totally unimodular $m \times n$ matrix with full row rank. Let B be a basis of A (that is, a regular $n \times n$ submatrix of A). Prove that $B^{-1}A$ is totally unimodular.

Problem 19. Prove that if a matrix A and a vector $\boldsymbol{b}$ are rational, then every vertex of the polyhedron $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}\}$ is rational. Next, prove that if every vertex of a polytope P is rational, then there exists rational A and $\boldsymbol{b}$ such that $P = \{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}\}$.

4 Cutting planes

Problem 20. Let $P = \operatorname{conv}\left\{(0,0),(1,0),(\frac{1}{2},3)\right\}$ and P' be the convex hull of all integral points of P. First, find a system of linear inequalities which determines P. Then, using Chvátal-Gomory cutting planes derive a system of linear inequalities which determines P'.

Try to generalize this approach for polyhedron $P = \operatorname{conv}\left\{(0,0),(1,0),(\frac{1}{2},k)\right\}$ where $k \in \mathbb{N}$.