ON HYPERCUBE 3-SPANNERS

PETR GREGOR

Abstract. A spanning subgraph S of a graph G is t-spanner if every two neighbors in G have distance at most t in S. We show that every 3-spanner of the n-dimensional hypercube Q_n has at least $(2 - o(1))2^n$ edges. On the other hand, there is a 3-spanner of Q_n with at most $3.5 \cdot 2^n$ edges. This improves previously known lower and upper bounds on the minimal size of hypercube 3-spanners. Furthermore, we present two constructions that are efficient for small dimensions.

1. Introduction

A spanning subgraph S of a graph G is *t-spanner* if every two neighbors in G have distance at most t in S. The parameter t is called *dilation* of the spanner.

Spanners were introduced by Peleg and Ullman [5] in 1989 for a synchronization of asynchronous distributed networks. Since then, they were found interesting for many other applications in interconnection networks and related areas; see for example [3] for further references.

For t-spanners of the n-dimensional hypercube Q_n, as for all bipartite graphs, the smallest non-trivial dilation t is 3. The study of hypercube 3-spanners was initiated also by Peleg and Ullman [5]. They constructed 3-spanners of Q_n for every dimension n with fewer than $7m$ edges where $m = |V(Q_n)| = 2^n$. Although this is asymptotically optimal, a determination of the minimal size of hypercube 3-spanners remains an open problem.

Duckworth and Zito [2] improved the construction from [5] and obtained 3-spanners of Q_n with at most $4m$ edges for every n. Moreover, for $n = 2^k - 2$ where k is an integer, they found 3-spanners of Q_n with at most $(3 - 2^{2-k})m$ edges. They also established a non-trivial lower bound $3mn/2(n + 3) = (3/2 - o(1))m$ on the minimal size of a 3-spanner of Q_n.

This paper continues with the investigation of the minimal size of a hypercube 3-spanner. Applying the method from [2] with some additional improvements, it is shown that Q_n has a 3-spanner with at most $3.5m$ edges for every n. On the other hand, we give an improved lower bound of $m(2n^2 - 5n - 1)/(n^2 + n - 10) = (2 - o(1))m$ edges in any 3-spanner S of Q_n. This bound is achieved by analyzing vertex degrees in S. Last but not least, two constructions that give efficient hypercube 3-spanners for small dimensions are presented. The latter one is based on two component 1-factors from the equitable 1-factorization of a Hamming shell by Dejter [1].

2. Preliminaries

The n-dimensional *hypercube* Q_n is a graph on all binary vectors of length n as vertices, and with edges joining vertices that differ in exactly one coordinate. It has $n2^{n-1}$ edges. Let $e_1, \dots, e_n$ denote the vertices with only one 1; on the i-th coordinate in e_i. An edge $\{u, v\} \in E(Q_n)$ has *direction* i if $u \oplus v = e_i$, where $u \oplus v$ is vector addition mod 2. For a set $A \subseteq V(Q_n)$, define $A \oplus e_i = \{u \oplus e_i \mid u \in A\}$. A *weight* $w(u)$ of a vertex $u \in V(Q_n)$ is the number of 1's in u. A vertex is *even* (respectively *odd*) if it has an even (respectively odd) weight. Even and odd vertices form bipartite sets of Q_n.

For any two vertices $u, v \in V(Q_n)$, the subgraph of Q_n induced by all shortest paths between u and v forms a subcube C of dimension equal to the distance of u and v. We call C the *subcube between u and v*.

For $n = r_1 + r_2$, the hypercube Q_n can be considered as a Cartesian product of hypercubes Q_{r_1} and Q_{r_2}. Every vertex $u \in V(Q_n)$ can be represented as a concatenation $u = xy$ of vertices

$x \in V(Q_{r_1})$ and $y \in V(Q_{r_2})$. For every $x \in V(Q_{r_1})$, there is a separate copy of Q_{r_2} in Q_n corresponding to the vertex x; similarly for $y \in V(Q_{r_2})$.

Let $d_G(u)$ be the degree of a vertex u in a graph G, let $d_G(u,v)$ be the distance of vertices u and v in G, and let $N_G(u)$ be the set of neighbors of u in G. The index referring to the underlying graph is omitted if it is clear from the context. A set $A \subseteq V(G)$ is a *dominating set* of G if every vertex is in A or has a neighbor in A. A dominating set A is *perfect* if every vertex is either in A or has a unique neighbor in A.

Let $\upsilon_r(i)$ denote the binary vector of length r that represents an integer $i < 2^r$. For $n = 2^r - 1$, consider the $r \times n$ matrix M_r whose columns are all the distinct nonzero binary vectors of length r. A *Hamming code* of length n [4] is the set of all binary vectors of length n that are orthogonal to all the rows of M_r. Suppose that the i-th column in M_r is $\upsilon_r(i)$. Then let H_n be the corresponding Hamming code. Other Hamming codes are isomorphic via a permutation of their coordinates.

It is well-known that H_n is a perfect one-error-correcting code; that is, a perfect dominating set in Q_n. It has size $2^n/(n+1)$ and minimal distance 3. For a pair $u, v \in H_n$ with $d(u,v) = 3$, consider the triple $\{i,j,k\}$ such that $u \oplus v = e_i \oplus e_j \oplus e_k$. Note that $u \oplus v \in H_n$, since H_n is a linear code. The set L_n of all such triples forms a *Steiner triple system*; that is, every pair $\{i,j\}$ is contained in exactly one triple. Moreover, $\{i,j,k\} \in L_n$ if and only if $\upsilon_r(i) \oplus \upsilon_r(j) \oplus \upsilon_r(k) = 0$.

The complement $\Sigma_n = Q_n \setminus H_n$ of the Hamming code H_n in the hypercube Q_n is called a *Hamming shell*.

Let $[n]$ denote the set $\{1, 2, \dots, n\}$.

3. Lower bound

We start with a local analysis of degrees in any hypercube 3-spanner.

Lemma 1. *Let S be a 3-spanner of Q_n. For any $u \in V(Q_n)$,*

$$\sum_{v \in N_S(u)} d_S(v) \geq n.$$

Proof. Let $A \subseteq E(S)$ be the set of edges incident with $N_S(u)$. Thus $|A| = \sum_{v \in N_S(u)} d_S(v)$. Suppose on the contrary that $|A| < n$. Then for some $i \in [n]$, the set A contains no edge with the direction i. Hence $d_S(u, u \oplus e_i) > 3$ and therefore S is not a 3-spanner of Q_n. □

We immediately obtain a global relation between the number of vertices with degree n and with degree 1.

Lemma 2. *A 3-spanner S of Q_n contains no more vertices with degree 1 than vertices with degree n.*

Proof. If a vertex u has degree 1, then his only neighbor v in S has degree n by Lemma 1. On the other hand, since S is a 3-spanner, any vertex v can be adjacent in S to at most one vertex u with degree 1. □

This allows us to establish the following bound on the sum of squares of all vertex degrees.

Lemma 3. *Let S be a 3-spanner of Q_n with s edges, $n \geq 4$. Then*

$$\sum_{u \in V(Q_n)} d_S^2(u) \leq \frac{2s(n^2-7) - 2^{n+1}(n^2-2n-1)}{n-3}.$$

Proof. Put $D = \sum_{u \in V(Q_n)} d_S^2(u)$ and let x and y be the numbers of vertices with degree n and degree 1 respectively, so $x \geq y$ by Lemma 2. We bound D over all distributions of degrees from $[n]$ such that $x \geq y$.

Since $\sum_{u \in V(Q_n)} d_S(u) = 2s$ is fixed, D is bounded by the case when x is maximal, $y = x$, $2^n - 2x - 1$ vertices have degree 2, and the remaining vertex has a degree r where $2 \leq r \leq n-2$. Then

$$2s = xn + 2^{n+1} - 4x - 2 + x + r = x(n-3) + 2^{n+1} - 2 + r$$

where $2 \le r \le n-2$, so

$$x = \left\lfloor \frac{2s - 2^{n+1}}{n-3} \right\rfloor, \; r = 2 + 2s - 2^{n+1} - x(n-3).$$

We have

$$D \le n^2 x + 4(2^n - 2x - 1) + x + r^2 = x(n^2 - 7) + 2^{n+2} + r^2 - 4.$$

To reduce r, put

$$x' = \frac{2s - 2^{n+1}}{n-3} \text{ or equivalently } x' = x + \frac{r-2}{n-3}.$$

Since

$$r^2 - 4 \le \frac{r-2}{n-3}(n^2 - 7) \text{ for } n \ge 4,$$

we conclude that

$$D \le x'(n^2 - 7) + 2^{n+2} = \frac{2s(n^2 - 7) - 2^{n+1}(n^2 - 2n - 1)}{n-3}.$$

□

The lower bound follows by counting the number of so called useful vertices.

Theorem 1. *Let S be a 3-spanner of Q_n with s edges. Then*

$$s \ge 2^n \frac{2n^2 - 5n - 1}{n^2 + n - 10} = 2^n(2 - o(1)).$$

Proof. We say that a vertex u is *useful* for an edge $\{x, y\} \in E(Q_n) \setminus E(S)$ if u belongs to a path in S of length 3 between x and y but $u \notin \{x, y\}$. Since S is a 3-spanner, every edge $\{x, y\} \in E(Q_n) \setminus E(S)$ has at least 2 useful vertices. On the other hand, every vertex u can be useful for at most $\binom{d_S(u)}{2}$ edges. So

$$\sum_{u \in V(Q_n)} \binom{d_S(u)}{2} \ge 2(|E(Q_n)| - s).$$

Since $\sum_{u \in V(Q_n)} d_S(u) = 2s$, we obtain

$$\sum_{u \in V(Q_n)} d_S^2(u) + 2s \ge n2^{n+1}.$$

Applying Lemma 3 and after few steps it follows that

$$s \ge 2^n \frac{2n^2 - 5n - 1}{n^2 + n - 10} \text{ for } n \ge 4,$$

whereas for $n \le 3$ this bound is tight, by a simple case-analysis. □

4. Upper bound

First we recall the method from [5]. Let D_1 and D_2 be dominating sets of hypercubes Q_{r_1} and Q_{r_2} respectively. For $n = r_1 + r_2$, we can construct a 3-spanner S of Q_n consisting of all the edges of the form

type (1): $\{xy, xy'\} \mid y \in D_2$ and $\{y, y'\} \in E(Q_{r_2})$,
type (2): $\{xy, x'y\} \mid x \in D_1$ and $\{x, x'\} \in E(Q_{r_1})$,
type (3): $\{xy, xy'\} \mid x \in D_1$ and $\{y, y'\} \in E(Q_{r_2})$,
type (4): $\{xy, x'y\} \mid y \in D_2$ and $\{x, x'\} \in E(Q_{r_1})$.

Note that every vertex $u = xy$ with $x \in D_1$ or $y \in D_2$ has degree n in S, while other vertices have degree at least 2 in S (equality holds if D_1 and D_2 are perfect). Also note that the edges emanating from a vertex xy with $x \in D_1$ and $y \in D_2$ are either both of type (1) and (3), or both of type (2) and (4).

To verify that S is a 3-spanner of Q_n, assume that we have an edge of the form $\{xy, xy'\} \in E(Q_n) \setminus E(S)$. Then $x \notin D_1$, otherwise it is an edge of type (3). Thus x has a neighbor $\bar{x} \in D_1$. Then $xy, \bar{x}y, \bar{x}y', xy'$ is an induced path in S of length 3. Similarly for an edge of the form $\{xy, x'y\} \in E(Q_n) \setminus E(S)$.

Applying this method on two hypercubes $Q_{n/2}$ with perfect dominating sets, we obtain a result from [2] which we restate with the proof.

Lemma 4. [2] *Let $n = 2^k - 2$ where k is an integer. The hypercube Q_n has a 3-spanner with $2^n(3 - 10/(n+2) + 8/(n+2)^2)$ edges.*[1]

Proof. Recall that for $r = n/2$, the hypercube Q_r has a perfect dominating set of size $2^r/(r+1)$. Let S be the 3-spanner of Q_n constructed above. Counting precisely the number of edges in S we have

$$\text{type (1) only: } \frac{r2^r}{r+1}\left(2^r - \frac{2^r}{r+1}\right), \text{ type (2) only: } \frac{r2^r}{r+1}\left(2^r - \frac{2^r}{r+1}\right),$$
$$\text{type (3): } \frac{r2^r2^{r-1}}{r+1}, \text{ type (4): } \frac{r2^r2^{r-1}}{r+1}.$$

Altogether we have

$$|E(S)| = \frac{2r2^r}{r+1}\left(2^r - \frac{2^r}{r+1}\right) + \frac{2r2^r2^{r-1}}{r+1} = 2^{2r}\frac{r(3r+1)}{(r+1)^2} =$$
$$= 2^{2r}\left(3 - \frac{5}{r+1} + \frac{2}{(r+1)^2}\right) = 2^n\left(3 - \frac{10}{n+2} + \frac{8}{(n+2)^2}\right).$$

□

It is worth stating that Lemma 4 can be slightly improved simply by omitting some edges.

Lemma 5. *Let $n = 2^k - 2$ where k is an integer. The hypercube Q_n has a 3-spanner with $2^n(3 - 14/(n+2) + 20/(n+2)^2)$ edges.*

Proof. Let S be the 3-spanner constructed by Lemma 4. Consider a vertex $u = xy$ with $x \in D_1$ and $y \in D_2$, where D_1 and D_2 are the dominating sets of Q_r used in the construction, $r = n/2$. Recall that $d_S(u) = n$ and $d_S(v) = n$ for any neighbor v of u.

After removing all but one edge emanating from u, we still have a 3-spanner S' of Q_n. Let $\{xy, xy'\} \in E(S)$, where $y' = y \oplus e_1$, be the edge that remains in S'. There are $2^r2^r/(r+1)^2$ such vertices $u = xy$, so

$$E(S') = E(S) - \frac{2^r2^r}{(r+1)^2}(n-1) = 2^{2r}\frac{3r^2 - r + 1}{(r+1)^2} =$$
$$= 2^{2r}\left(3 - \frac{7}{r+1} + \frac{5}{(r+1)^2}\right) = 2^n\left(3 - \frac{14}{n+2} + \frac{20}{(n+2)^2}\right).$$

□

Let $D_S = \{xy \mid x \in V(Q_r),\ y \in D_2\}$ and $D_{S'} = D_S \setminus \{xy \mid x \in D_1,\ y \in D_2\} \cup \{xy' \mid x \in D_1,\ y \in D_2,\ y' = y \oplus e_1\}$. Sets D_S and $D_{S'}$ are dominating sets of 3-spanners S and S'. They both have $2^n/(r+1) = 2^{n+1}/(n+2)$ vertices.

The key idea in the further improvement is to apply the same method also for hypercubes Q_{r_1} and Q_{r_2} with perfect dominating sets but different dimensions r_1 and r_2.

Lemma 6. *Let $n = 3.2^k - 2$ where k is an integer. The hypercube Q_n has a 3-spanner with $2^{n-2}(13 - 54/(n+2) + 63/(n+2)^2)$ edges.*

[1]In the original paper [2] it was "at most $2^n(3-4/(n+2))$" which follows from the fact that $8/(n+2)^2 \leq 6/(n+2)$ for all $n \geq 1$.

Proof. Put $r_1 = 2^k - 1 = (n-1)/3$ and $r_2 = 2^{k+1} - 1 = 2r_1 + 1 = (2n+1)/3$. Hypercubes Q_{r_1} and Q_{r_2} have perfect dominating sets D_1 and D_2 of sizes $2^{r_1}/(r_1+1)$ and $2^{r_2}/(r_2+1) = 2^{2r_1}/(r_1+1)$ respectively. Let $\bar{S}$ be the 3-spanner of Q_n obtained from the construction above. Counting the number of edges we have

$$\text{type (1) only: } \frac{r_2 2^{r_2}}{r_2+1}\left(2^{r_1} - \frac{2^{r_1}}{r_1+1}\right) = \frac{r_1(2r_1+1)2^{3r_1}}{(r_1+1)^2},$$
$$\text{type (2) only: } \frac{r_1 2^{r_1}}{r_1+1}\left(2^{r_2} - \frac{2^{r_2}}{r_2+1}\right) = \frac{r_1(2r_1+1)2^{3r_1}}{(r_1+1)^2},$$
$$\text{type (3): } \frac{r_2 2^{r_1} 2^{r_2-1}}{r_2+1} = \frac{(2r_1+1)2^{3r_1}}{r_1+1}, \text{ type (4): } \frac{r_1 2^{r_2} 2^{r_1-1}}{r_2+1} = \frac{r_1 2^{3r_1-1}}{r_1+1}.$$

As in Lemma 5, we can remove $2^{r_1}2^{r_2}(n-1)/((r_1+1)(r_2+1)) = 3r_1 2^{3r_1}/(r_1+1)^2$ edges. Altogether we have

$$\begin{aligned} E(\bar{S}) &= \frac{2r_1(2r_1+1)2^{3r_1}}{(r_1+1)^2} + \frac{(2r_1+1)2^{3r_1}}{r_1+1} + \frac{r_1 2^{3r_1-1}}{r_1+1} - \frac{3r_1 2^{3r_1}}{(r_1+1)^2} = \\ &= 2^{3r_1-1}\frac{13{r_1}^2 + 8r_1 + 2}{(r_1+1)^2} = 2^{3r_1-1}\left(13 - \frac{18}{r_1+1} + \frac{7}{(r_1+1)^2}\right) = \\ &= 2^{n-2}\left(13 - \frac{54}{n+2} + \frac{63}{(n+2)^2}\right). \end{aligned}$$

□

Let $D_{\bar{S}}$ be the dominating set of $\bar{S}$ obtained in a similar way as $D_{S'}$. It has $2^{r_1}2^{r_2}/(r_2+1) = 3.2^{n-1}/(n+2)$ vertices.

Now we combine previous two lemmas with the method from [2] to construct hypercube 3-spanners for general dimensions.

Consider the hypercube Q_n as a Cartesian product of Q_r and Q_{n-r} where $r < n$. Assume that we have a 3-spanner S_r of Q_r with a dominating set D. Then we can construct a 3-spanner S_n of Q_n by taking S_r in each copy of Q_r and by adding, for each $x \in D$, a full copy of Q_{n-r} corresponding to the vertex x.

To verify that S_n is a 3-spanner of Q_n, assume first that we have an edge of the form $\{xy, x'y\} \in E(Q_n) \setminus E(S_n)$. Then there is an induced path of length 3 between xy and $x'y$ in a copy of S_r in the hypercube Q_r corresponding to the vertex y. Secondly, for an edge of the form $\{xy, xy'\} \in E(Q_n) \setminus E(S_n)$, we have $x \notin D$. Thus x has a neighbor $\bar{x} \in D$. Then $xy, \bar{x}y, \bar{x}y', xy'$ is an induced path in S_n of length 3.

Theorem 2. *For every $n \geq 1$, the hypercube Q_n has a 3-spanner with at most $3.5 \cdot 2^n$ edges.*

Proof. Let $k = \lfloor \log_2(n+2) \rfloor$, so $2^k - 2 \leq n < 2^{k+1} - 2$. We have two cases.

(1) $2^k - 2 \leq n < 3.2^{k-1} - 2$. Let $r = 2^k - 2$ so $n - r \leq 2^{k-1} - 1$. Applying Lemma 5 and the idea above we obtain a 3-spanner S of Q_n of size

$$\begin{aligned} |E(S)| &= |E(S')| \cdot 2^{n-r} + |D_{S'}| \cdot |E(Q_{n-r})| = \\ &= 2^r\left(3 - \frac{14}{r+2} + \frac{20}{(r+2)^2}\right)2^{n-r} + \frac{2^{r+1}}{r+2}(n-r)2^{n-r-1} \\ &\leq 2^n\left(3.5 - \frac{15}{r+2} + \frac{20}{(r+2)^2}\right) \text{ since } \frac{n-r}{r+2} \leq \frac{1}{2} - \frac{1}{r+2}, \\ &\leq 3.5 \cdot 2^n. \end{aligned}$$

(2) $3.2^{k-1}-2 \leq n < 2^{k+1}-2$. Let $r = 3.2^{k-1}-2$ so $n-r \leq 2^{k-1}-1$. Applying Lemma 6 and the idea above we obtain a 3-spanner S of Q_n of size

$$\begin{aligned}|E(S)| &= |E(\bar{S})| \cdot 2^{n-r} + |D_{\bar{S}}| \cdot |E(Q_{n-r})| = \\ &= 2^{r-2}\left(13 - \frac{54}{r+2} + \frac{63}{(r+2)^2}\right)2^{n-r} + \frac{3.2^{r-1}}{r+2}(n-r)2^{n-r-1} \\ &\leq 2^{n-2}\left(14 - \frac{57}{r+2} + \frac{63}{(r+2)^2}\right) \text{ since } \frac{n-r}{r+2} \leq \frac{1}{3} - \frac{1}{r+2}, \\ &\leq 3.5 \cdot 2^n.\end{aligned}$$

□

5. Construction for small dimension

In this section we present two constructions that give efficient hypercube 3-spanners for small dimension. This is motivated by the fact that the dimension is bounded in most hypercube applications. However, these constructions use more than a linear number of edges with respect to the number of vertices.

Lemma 7. *For $n \geq 3$, the hypercube Q_n has a 3-spanner with $2^n(n+5)/8$ edges.*

Proof. For $n = 3$, let $A \subseteq E(Q_3)$ be the set of all edges incident with the Hamming code $H_3 = \{000, 111\}$. Observe that the set $E(S) = A \cup \{\{010, 110\}, \{001, 101\}\}$ forms a 3-spanner S of Q_3 of size 8.

For $n > 3$, the hypercube Q_n can be considered as a Cartesian product of Q_3 and Q_{n-3}. In each copy of Q_3 take a 3-spanner S and add two full copies of Q_{n-3} corresponding to vertices in H_3. We obtain a 3-spanner of Q_n of size

$$8.2^{n-3} + 2(n-3)2^{n-4} = 2^n \frac{n+5}{8}.$$

□

Lemma 8. *For $n \geq 7$, the hypercube Q_n has a 3-spanner with $2^n(n+21)/16$ edges.*

Proof. For $n = 7$, we have the Hamming code and the Steiner triple system:

$$\begin{aligned}H_7 = \{&1110000, 0101010, 0010110, 0011001, 1001100, 1000011, 0100101, 0000000, \\ &0001111, 1010101, 1101001, 1100110, 0110011, 0111100, 1011010, 1111111\}, \\ L_7 = \{&\{1,2,3\}, \{2,4,6\}, \{3,5,6\}, \{3,4,7\}, \{1,4,5\}, \{1,6,7\}, \{2,5,7\}\}.\end{aligned}$$

Define permutations s and z of $[7]$ by the following table

$$\begin{aligned}i &= 1, 2, 3, 4, 5, 6, 7, \\ s(i) &= 2, 4, 6, 3, 1, 7, 5, \\ z(i) &= 3, 6, 5, 7, 4, 1, 2.\end{aligned}$$

Note that $L_n = \{\{i, s(i), z(i)\} \mid i \in [7]\}$. Let A be the set of all edges incident with the Hamming code H_n. For $i \in [7]$ define sets

$$B_i = \{\{u \oplus e_{s(i)}, u \oplus e_{s(i)} \oplus e_i\} \in E(Q_7) \mid u \in H_7\},$$

i.e. B_i is the set of edges with direction i from $H_n \oplus e_{s(i)}$. Observe that we could replace $s(i)$ with $z(i)$ in the definition of B_i since $H_n \oplus e_i \oplus e_{s(i)} \oplus e_{z(i)} = H_n$ for all $i \in [7]$. Put $B = \bigcup_{i=1}^{n} B_i$. We claim that the set $E(S') = A \cup B$ forms a 3-spanner S' of Q_7 of size 224.

Assume that we have an edge $\{x, y\} \in E(Q_7) \setminus A$. It follows that $x, y \notin H_7$. Let u and v be the neighbors from H_7 of x and y respectively. Since $d_{Q_7}(u, v) = 3$, there is a unique $i \in [7]$ such that $u \oplus v = e_i \oplus e_{s(i)} \oplus e_{z(i)}$. Denote by C the subcube of dimension 3 in Q_7 between u and v, and consider the subgraph $S' \cap C$. It contains edges

$$E(A \cap C) = \{\{w, w \oplus e_j\} \mid w \in \{u, v\},\ j \in \{i, s(i), z(i)\}\},\ \text{and}$$
$$E(B_i \cap C) = \{\{u \oplus e_{s(i)}, u \oplus e_{s(i)} \oplus e_i\}, \{v \oplus e_{s(i)}, v \oplus e_{s(i)} \oplus e_i\}\}.$$

Observe that $S' \cap C$ is isomorphic to the 3-spanner S of Q_3 from Lemma 7. We conclude that $d_{S'}(x, y) = d_S(x, y) \leq 3$ since $x, y \in V(C)$. Counting the number of edges we have $|E(S')| = 14|H_7| = 224$.

For $n > 7$, the hypercube Q_n can be considered as a Cartesian product of Q_7 and Q_{n-7}. In each copy of Q_7 take a 3-spanner S' and add 16 full copies of Q_{n-7} corresponding to vertices in H_7. We obtain a 3-spanner of Q_n of size

$$224.2^{n-7} + 16(n-7)2^{n-8} = 2^n \frac{n+21}{16}.$$

□

6. Conclusions

Comparing Lemma 7, Lemma 8 and the construction in Theorem 2, we obtain that 3-spanners of Q_n with the fewest edges are provided by Lemma 7 when $3 \leq n \leq 11$, by Lemma 8 when $11 \leq n \leq 22$, and by Theorem 2 when $n \geq 23$. Note that when $n = 11$, both Lemma 7 and Lemma 8 give 3-spanners of Q_n with 2^{12} edges.

The set B from the proof of Lemma 8 is a 2-factor of the Hamming shell Σ_7. It consists of two 1-factors $B^1 = \bigcup_{i=1}^n B_i^1$ and $B^2 = \bigcup_{i=1}^n B_i^2$ where

$$B_i^1 = \{\{u \oplus e_{s(i)}, u \oplus e_{s(i)} \oplus e_i\} \in E(Q_n) \mid u \in H_n,\ u \text{ is even}\},$$
$$B_i^2 = \{\{u \oplus e_{z(i)}, u \oplus e_{z(i)} \oplus e_i\} \in E(Q_n) \mid u \in H_n,\ u \text{ is even}\}.$$

The sets B^1 and B^2 are component 1-factors from the equitable 1-factorization of Hamming shells given by Dejter [1] for all $n = 2^k - 1$, where k is an integer. Lemma 8 could be generalized also for $n = 2^k - 1$ with $k > 3$ by adding to the set B more 1-factors from this 1-factorization. However, the Steiner triple system L_n contains $n(n-1)/6$ triples, so we would need at least $(n-1)/3$ 1-factors which would result into an inefficient 3-spanner.

Although a further improvement of the general upper bound seems difficult with the methods used so far, a new construction may arise from the closely related coding theory. On the other hand, exploring larger neighborhoods or counting the number of vertices of small degrees may improve the lower bound.

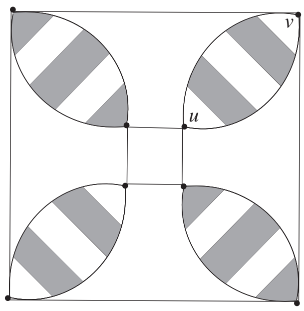

Figure 1. A 3-spanner of Q_7 that is not an additive 4-spanner. The edges between the copies of Q_5 are all in the spanner, but are omitted in the picture, except the edges from vertices u and v. The gray areas represent all edges between two consecutive levels of Q_5.

A spanner S of a graph G has the *delay c* if $d_S(u, v) \leq d_G(u, v) + c$ for any pair of vertices u and v. Such spanners are called *additive c-spanners*. It is not difficult to verify that all hypercube 3-spanners constructed in this paper are actually additive 4-spanners. This does not hold in general.

An example of a 3-spanner of Q_7 that is not an additive 4-spanner is depicted on Figure 1. The vertices u and v have $d_{Q_7}(u,v) = 5$ but $d_S(u,v) = 11$.

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Department of Theoretical Computer Science and Mathematical Logic, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
E-mail address: gregor@ktiml.mff.cuni.cz