Propositional and Predicate Logic - VI and $\frac{1}{2}$

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Basic algebraic theories

- theory of *groups* in the language $L = \langle +, -, 0 \rangle$ with equality has axioms

  $x + (y + z) = (x + y) + z$ (associativity of $+$)

  $0 + x = x = x + 0$ (0 is neutral to $+$)

  $x + (-x) = 0 = (-x) + x$ ($-x$ is inverse of x)

- theory of *Abelian groups* has moreover ax. $x + y = y + x$ (commutativity)

- theory of *rings* in $L = \langle +, -, \cdot, 0, 1 \rangle$ with equality has moreover axioms

  $1 \cdot x = x = x \cdot 1$ (1 is neutral to $\cdot$)

  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity of $\cdot$)

  $x \cdot (y + z) = x \cdot y + x \cdot z,\ (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity)

- theory of *commutative rings* has moreover ax. $x \cdot y = y \cdot x$ (commutativity)

- theory of *fields* in the same language has additional axioms

  $x \neq 0 \to (\exists y)(x \cdot y = 1)$ (existence of inverses to $\cdot$)

  $0 \neq 1$ (nontriviality)

Properties of theories

A theory T of a language L is *(semantically)*

- *inconsistent* if $T \models \bot$, otherwise T is *consistent* (*satisfiable*),
- *complete* if it is consistent and every sentence of L is valid in T or contradictory in T, i.e. no sentence is independent in T.

Structures $\mathcal{A}$, $\mathcal{B}$ for a language L are *elementarily equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same sentences of L. For example, $\langle \mathbb{Q}, \leq \rangle \equiv \langle \mathbb{R}, \leq \rangle$.

Observation *A theory T is (semantically)*

(1) *consistent if and only if it has a model,*

(2) *complete if and only if it has exactly one model, up to elementary equivalence, i.e. all its models are elementarily equivalent.*

Substructures

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ and $\mathcal{B} = \langle B, \mathcal{R}^B, \mathcal{F}^B \rangle$ be structures for $L = \langle \mathcal{R}, \mathcal{F} \rangle$.

We say that $\mathcal{B}$ is an (induced) *substructure* of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if

(i) $B \subseteq A$,

(ii) $R^B = R^A \cap B^{\mathrm{ar}(R)}$ for every $R \in \mathcal{R}$,

(iii) $f^B = f^A \cap (B^{\mathrm{ar}(f)} \times B)$; that is, $f^B = f^A \restriction B^{\mathrm{ar}(f)}$, for every $f \in \mathcal{F}$.

A set $C \subseteq A$ is a domain of some substructure of $\mathcal{A}$ if and only if C is closed under all functions of $\mathcal{A}$. Then the respective substructure, denoted by $\mathcal{A} \restriction C$, is said to be the *restriction* of the structure $\mathcal{A}$ to C.

- A set $C \subseteq A$ is *closed* under a function $f \colon A^n \to A$ if $f(x_1, \ldots, x_n) \in C$ for every $x_1, \ldots, x_n \in C$.

Example: $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$ *is a substructure of* $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$ *and* $\underline{\mathbb{Z}} = \underline{\mathbb{Q}} \restriction \mathbb{Z}$. *Furthermore,* $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *is their substructure and* $\underline{\mathbb{N}} = \underline{\mathbb{Q}} \restriction \mathbb{N} = \underline{\mathbb{Z}} \restriction \mathbb{N}$.

Validity in a substructure

Let $\mathcal{B}$ be a substructure of a structure $\mathcal{A}$ for a (fixed) language L.

Proposition *For every open formula φ and assignment $e\colon \mathrm{Var} \to B$,*

$$\mathcal{A} \models \varphi[e] \quad \textit{if and only if} \quad \mathcal{B} \models \varphi[e].$$

Proof For atomic φ it follows from the definition of the truth value with respect to an assignment. Otherwise by induction on the structure of the formula. □

Corollary *For every open formula φ and structure $\mathcal{A}$,*

$$\mathcal{A} \models \varphi \quad \textit{if and only if} \quad \mathcal{B} \models \varphi \textit{ for every substructure } \mathcal{B} \subseteq \mathcal{A}.$$

- A theory T is *open* if all axioms of T are open.

Corollary *Every substructure of a model of an open theory T is a model of T.*

For example, every substructure of a graph, i.e. a model of theory of graphs, is a graph, called a subgraph. Similarly subgroups, but not subfields.

Generated substructure, expansion, reduct

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be a structure and $X \subseteq A$. Let B be the smallest subset of A containing X that is closed under all functions of the structure $\mathcal{A}$ (including constants). Then the structure $\mathcal{A} \restriction B$ is denoted by $\mathcal{A}\langle X \rangle$ and is called the substructure of $\mathcal{A}$ *generated* by the set X.

Example: for $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$, $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$, $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *it is* $\underline{\mathbb{Q}}\langle \{1\} \rangle = \underline{\mathbb{N}}$, $\underline{\mathbb{Q}}\langle \{-1\} \rangle = \underline{\mathbb{Z}}$, *and* $\underline{\mathbb{Q}}\langle \{2\} \rangle$ *is the substructure on all even natural numbers.*

Let $\mathcal{A}$ be a structure for a language L and $L' \subseteq L$. By omitting realizations of symbols that are not in L' we obtain from $\mathcal{A}$ a structure $\mathcal{A}'$ called the *reduct* of $\mathcal{A}$ to the language L'. Conversely, $\mathcal{A}$ is an *expansion* of $\mathcal{A}'$ into L.

For example, $\langle \mathbb{N}, + \rangle$ *is a reduct of* $\langle \mathbb{N}, +, \cdot, 0 \rangle$. *On the other hand, the structure* $\langle \mathbb{N}, +, c_i \rangle_{i \in \mathbb{N}}$ *with* $c_i = i$ *for every* $i \in \mathbb{N}$ *is the expansion of* $\langle \mathbb{N}, + \rangle$ *by names of elements from* $\mathbb{N}$.

Theorem on constants

Theorem *Let φ be a formula in a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

$$T \models \varphi \quad \textit{if and only if} \quad T' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

Proof ($\Rightarrow$) If $\mathcal{A}'$ is a model of T', let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \varphi[e]$ for every assignment e, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{\mathcal{A}'}, \ldots, x_n/c_n^{\mathcal{A}'})], \quad \text{i.e.} \quad \mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

($\Leftarrow$) If $\mathcal{A}$ is a model of T and e an assignment, let $\mathcal{A}'$ be the expansion of A into L' by setting $c_i^{\mathcal{A}'} = e(x_i)$ for every i. Since $\mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n)[e']$ for every assignment e', we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{\mathcal{A}'}, \ldots, x_n/c_n^{\mathcal{A}'})], \quad \text{i.e.} \quad \mathcal{A} \models \varphi[e]. \quad \square$$

Extensions of theories

Let T and T' be theories of language L and L', respectively. We say that

- T' is an *extension* of T if $L \subseteq L'$ and $\theta^L(T) \subseteq \theta^{L'}(T')$,
- an extension T' of T is *simple* if $L = L'$,
- an extension T' of T is *conservative* if $\theta^L(T) = \theta^L(T') = \theta^{L'}(T') \cap \mathrm{Fm_L}$,
- T and T' are *equivalent* if T' is an extension of T and vice-versa.

Observation *Let T and T' be theories over L and L', and $L \subseteq L'$. Then*

(1) *T' is a simple extension of T if and only if $L = L'$ and $M^L(T') \subseteq M^L(T)$,*

(2) *T' is an extension of T if and only if the reduct of every model of T' to the language L is a model of T,*

(3) *T' is a conservative extension of T if and only if T' is an extension of T and every model of T can be expanded into some model of T',*

(4) *T' and T are equivalent if and only if $L = L'$ and $M^L(T') = M^L(T)$.*