

Propositional and Predicate Logic - II

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Semantic notions

A proposition φ over $\mathbb{P}$ is

- *is true in* (*satisfied by*) *an assignment* $v: \mathbb{P} \rightarrow \{0, 1\}$, if $\bar{v}(\varphi) = 1$.
Then v is a *satisfying assignment* for φ , denoted by $v \models \varphi$.
- *valid* (*a tautology*), if $\bar{v}(\varphi) = 1$ for every $v: \mathbb{P} \rightarrow \{0, 1\}$,
i.e. φ is satisfied by every assignment, denoted by $\models \varphi$.
- *unsatisfiable* (*a contradiction*), if $\bar{v}(\varphi) = 0$ for every $v: \mathbb{P} \rightarrow \{0, 1\}$, i.e.
 $\neg\varphi$ is valid.
- *independent* (*a contingency*), if $\bar{v}_1(\varphi) = 0$ and $\bar{v}_2(\varphi) = 1$ for some
 $v_1, v_2: \mathbb{P} \rightarrow \{0, 1\}$, i.e. φ is neither a tautology nor a contradiction.
- *satisfiable*, if $\bar{v}(\varphi) = 1$ for some $v: \mathbb{P} \rightarrow \{0, 1\}$, i.e. φ is not a contradiction.

Propositions φ and ψ are (logically) *equivalent*, denoted by $\varphi \sim \psi$, if
 $\bar{v}(\varphi) = \bar{v}(\psi)$ for every $v: \mathbb{P} \rightarrow \{0, 1\}$, i.e. the proposition $\varphi \leftrightarrow \psi$ is valid.

Models

We reformulate these semantic notions in the terminology of models.

A **model of a language** $\mathbb{P}$ is a truth assignment of $\mathbb{P}$. The class of all models of $\mathbb{P}$ is denoted by $M(\mathbb{P})$. A proposition φ over $\mathbb{P}$ is

- **true in a model** $v \in M(\mathbb{P})$, if $\bar{v}(\varphi) = 1$. Then v is a **model of** φ , denoted by $v \models \varphi$ and $M^{\mathbb{P}}(\varphi) = \{v \in M(\mathbb{P}) \mid v \models \varphi\}$ is the **class of all models** of φ .
- **valid (a tautology)** if it is true in every model of the language, denoted by $\models \varphi$.
- **unsatisfiable (a contradiction)** if it does not have a model.
- **independent (a contingency)** if it is true in some model and false in other.
- **satisfiable** if it has a model.

Propositions φ and ψ are (logically) **equivalent**, denoted by $\varphi \sim \psi$, if they have same models.

Theory

Informally, a theory is a description of “world” to which we restrict ourselves.

- A propositional *theory* over the language $\mathbb{P}$ is any set T of propositions from $\text{PF}_{\mathbb{P}}$. We say that propositions of T are *axioms* of the theory T .
- A *model of theory* T over $\mathbb{P}$ is an assignment $v \in M(\mathbb{P})$ (i.e. a model of the language) in which all axioms of T are true, denoted by $v \models T$.
- A *class of models* of T is $M^{\mathbb{P}}(T) = \{v \in M(\mathbb{P}) \mid v \models \varphi \text{ for every } \varphi \in T\}$.

For example, for $T = \{p, \neg p \vee \neg q, q \rightarrow r\}$ over $\mathbb{P} = \{p, q, r\}$ we have

$$M^{\mathbb{P}}(T) = \{(1, 0, 0), (1, 0, 1)\}$$

- If a theory is finite, it can be replaced by a *conjunction* of its axioms.
- We write $M(T, \varphi)$ as a shortcut for $M(T \cup \{\varphi\})$.

Semantics with respect to a theory

Semantic notions can be defined with respect to a theory, more precisely, with respect to its models. Let T be a theory over $\mathbb{P}$. A proposition φ over $\mathbb{P}$ is

- *valid in T* (*true in T*) if it is true in every model of T , denoted by $T \models \varphi$.
We also say that φ is a (semantic) *consequence* of T .
- *unsatisfiable* (*contradictory*) *in T* (*inconsistent with T*) if it is false in every model of T ,
- *independent (or contingency) in T* if it is true in some model of T and false in some other,
- *satisfiable in T* (*consistent with T*) if it is true in some model of T .

Propositions φ and ψ are *equivalent in T* (*T -equivalent*), denoted by $\varphi \sim_T \psi$, if for every model v of T , $v \models \varphi$ if and only if $v \models \psi$.

Note If all axioms of a theory T are valid (tautologies), e.g. for $T = \emptyset$, then all notions with respect to T correspond to the same notions in (pure) logic.

Universality

The language of propositional logic has *basic* connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. In general, we can introduce n -ary connective for any Boolean function, e.g.

$p \downarrow q$ “neither p nor q ” (NOR, Peirce arrow)

$p \uparrow q$ “not both p and q ” (NAND, Sheffer stroke)

A set of connectives is *universal* if every Boolean function can be expressed as a proposition formed from these connectives.

Proposition $\{\neg, \wedge, \vee\}$ is universal.

Proof A function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is expressed by $\bigvee_{v \in f^{-1}[1]} \bigwedge_{i=1}^n p_i^{v(i)}$

where $p_i^{v(i)}$ denotes the proposition p_i if $v(i) = 1$; and $\neg p_i$ if $v(i) = 0$.

For $f^{-1}[1] = \emptyset$ we take the proposition $\perp$. $\square$

Proposition $\{\neg, \rightarrow\}$ is universal.

Proof $(p \wedge q) \sim \neg(p \rightarrow \neg q)$, $(p \vee q) \sim (\neg p \rightarrow q)$. $\square$

CNF and DNF

- A **literal** is a propositional letter or its negation. Let p^1 be the literal p and let p^0 be the literal $\neg p$. Let $\bar{l}$ denote the **complementary** literal to a literal l .
- A **clause** is a disjunction of literals, by the **empty clause** we mean $\perp$.
- A proposition is in **conjunctive normal form (CNF)** if it is a conjunction of clauses. By the **empty proposition in CNF** we mean $\top$.
- An **elementary conjunction** is a conjunction of literals, by the **empty conjunction** we mean $\top$.
- A proposition is in **disjunctive normal form (DNF)** if it is a disjunction of elementary conjunctions. By the **empty proposition in DNF** we mean $\perp$.

Note A clause or an elementary conjunction is both in CNF and DNF.

Observation *A proposition in CNF is valid if and only if each of its clauses contains a pair of complementary literals. A proposition in DNF is satisfiable if and only if at least one of its elementary conjunctions does not contain a pair of complementary literals.*

Transformations by tables

Proposition Let $K \subseteq \{0, 1\}^{\mathbb{P}}$ where $\mathbb{P}$ is finite and $\overline{K} = \{0, 1\}^{\mathbb{P}} \setminus K$. Then

$$M^{\mathbb{P}}\left(\bigvee_{v \in K} \bigwedge_{p \in \mathbb{P}} p^{v(p)}\right) = K = M^{\mathbb{P}}\left(\bigwedge_{v \in \overline{K}} \bigvee_{p \in \mathbb{P}} \overline{p^{v(p)}}\right)$$

Proof The first equality follows from $w(\bigwedge_{p \in \mathbb{P}} p^{v(p)}) = 1$ if and only if $w = v$. Similarly, the second one follows from $w(\bigvee_{p \in \mathbb{P}} \overline{p^{v(p)}}) = 1$ if and only if $w \neq v$.

□

For example, $K = \{(1, 0, 0), (1, 1, 0), (0, 1, 0), (1, 1, 1)\}$ can be modeled by

$$(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \sim \\ (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$$

Corollary Every proposition has CNF and DNF equivalents.

Proof The value of a proposition φ depends only on the assignment of $\text{var}(\varphi)$ which is finite. Hence we can apply the above proposition for $K = M^{\mathbb{P}}(\varphi)$ and $\mathbb{P} = \text{var}(\varphi)$. □

Transformations by rules

Proposition Let φ' be the proposition obtained from φ by replacing some occurrences of a subformula ψ with ψ' . If $\psi \sim \psi'$, then $\varphi \sim \varphi'$.

Proof By induction on the structure of the formula. $\square$

$$(1) \quad (\varphi \rightarrow \psi) \sim (\neg\varphi \vee \psi), \quad (\varphi \leftrightarrow \psi) \sim ((\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi))$$

$$(2) \quad \neg\neg\varphi \sim \varphi, \quad \neg(\varphi \wedge \psi) \sim (\neg\varphi \vee \neg\psi), \quad \neg(\varphi \vee \psi) \sim (\neg\varphi \wedge \neg\psi)$$

$$(3) \quad (\varphi \vee (\psi \wedge \chi)) \sim ((\psi \wedge \chi) \vee \varphi) \sim ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$$

$$(3)' \quad (\varphi \wedge (\psi \vee \chi)) \sim ((\psi \vee \chi) \wedge \varphi) \sim ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$$

Proposition Every proposition can be transformed into CNF / DNF applying the transformation rules (1), (2), (3)/(3)'.

Proof By induction on the structure of the formula. $\square$

Proposition Assume that φ contains only $\neg, \wedge, \vee$ and φ^* is obtained from φ by interchanging $\wedge$ and $\vee$, and by complementing all literals. Then $\neg\varphi \sim \varphi^*$.

Proof By induction on the structure of the formula. $\square$

Consequence of a theory

The *consequence* of a theory T over $\mathbb{P}$ is the set $\theta^{\mathbb{P}}(T)$ of all propositions that are valid in T , i.e.

$$\theta^{\mathbb{P}}(T) = \{\varphi \in \text{PF}_{\mathbb{P}} \mid T \models \varphi\}.$$

Proposition For every theories $T \subseteq T'$ and propositions $\varphi, \varphi_1, \dots, \varphi_n$ over $\mathbb{P}$,

- (1) $T \subseteq \theta^{\mathbb{P}}(T) = \theta^{\mathbb{P}}(\theta^{\mathbb{P}}(T))$,
- (2) $T \subseteq T' \Rightarrow \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(T')$,
- (3) $\varphi \in \theta^{\mathbb{P}}(\{\varphi_1, \dots, \varphi_n\}) \Leftrightarrow \models (\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \varphi$.

Proof Easily from definitions, since $T \models \varphi \Leftrightarrow M(T) \subseteq M(\varphi)$ and

- (1) $M(\theta(T)) = M(T)$,
- (2) $T \subseteq T' \Rightarrow M(T') \subseteq M(T)$,
- (3) $\models \psi \rightarrow \varphi \Leftrightarrow M(\psi) \subseteq M(\varphi), M(\varphi_1 \wedge \dots \wedge \varphi_n) = M(\varphi_1, \dots, \varphi_n)$. □

Properties of theories

A propositional theory T over $\mathbb{P}$ is (*semantically*)

- *inconsistent* (*unsatisfiable*) if $T \models \perp$, otherwise is *consistent* (*satisfiable*),
- *complete* if it is consistent, and $T \models \varphi$ or $T \models \neg\varphi$ for every $\varphi \in \text{PF}_{\mathbb{P}}$,
i.e. no proposition over $\mathbb{P}$ is independent in T ,
- an *extension* of a theory T' over $\mathbb{P}'$ if $\mathbb{P}' \subseteq \mathbb{P}$ and $\theta^{\mathbb{P}'}(T') \subseteq \theta^{\mathbb{P}}(T)$;
we say that an extension T of a theory T' is *simple* if $\mathbb{P} = \mathbb{P}'$; and
conservative if $\theta^{\mathbb{P}'}(T') = \theta^{\mathbb{P}}(T) \cap \text{PF}_{\mathbb{P}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa,

Observation Let T and T' be theories over $\mathbb{P}$. Then T is (semantically)

- (1) *consistent if and only if it has a model,*
- (2) *complete if and only if it has a single model,*
- (3) *extension of T' if and only if $M^{\mathbb{P}}(T) \subseteq M^{\mathbb{P}}(T')$,*
- (4) *equivalent with T' if and only if $M^{\mathbb{P}}(T) = M^{\mathbb{P}}(T')$.*