Propositional and Predicate Logic - VI

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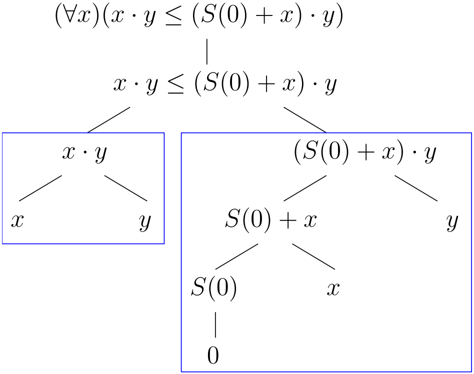
An example of a formula



The formation tree of the formula $(\forall x)(x \cdot y \leq (S(0) + x) \cdot y)$.

Occurrences of variables

Let φ be a formula and x be a variable.

- An *occurrence* of x in φ is a leaf labeled by x in the formation tree of φ.
- An occurrence of x in φ is *bound* if it is in some subformula ψ that starts with $(\forall x)$ or $(\exists x)$. An occurrence of x in φ is *free* if it is not bound.
- A variable x is *free* in φ if it has at least one free occurrence in φ. It is *bound* in φ if it has at least one bound occurrence in φ.
- A variable x can be both free and bound in φ. For example in
$$(\forall x)(\exists y)(x \leq y) \vee x \leq z.$$
- We write $\varphi(x_1, \dots, x_n)$ to denote that $x_1, \dots, x_n$ are all free variables in the formula φ. *(φ states something about these variables.)*

Remark *We will see that the truth value of a formula (in a given interpretation of symbols) depends only on the assignment of free variables.*

Open and closed formulas

- A formula is *open* if it is without quantifiers. For the set OFm_L of all open formulas in a language L it holds that $\mathrm{AFm}_L \subsetneq \mathrm{OFm}_L \subsetneq \mathrm{Fm}_L$.
- A formula is *closed* (a *sentence*) if it has no free variable; that is, all occurrences of variables are bound.
- A formula can be both open and closed. In this case, all its terms are ground terms.

$x + y \leq 0$	*open*, $\varphi(x, y)$
$(\forall x)(\forall y)(x + y \leq 0)$	*a sentence*,
$(\forall x)(x + y \leq 0)$	*neither open nor a sentence*, $\varphi(y)$
$1 + 0 \leq 0$	*open sentence*

Remark *We will see that in a fixed interpretation of symbols a sentence has a fixed truth value; that is, it does not depend on the assignment of variables.*

Instances

After substituting a term t for a free variable x in a formula φ, we would expect that the new formula (newly) says about t "the same" as φ did about x.

$\varphi(x)$	$(\exists y)(x+y=1)$	*"there is an element $1-x$"*
for $t=1$ we can $\varphi(x/t)$	$(\exists y)(1+y=1)$	*"there is an element $1-1$"*
for $t=y$ we cannot	$(\exists y)(y+y=1)$	*"1 is divisible by 2"*

- A term t is *substitutable* for a variable x in a formula φ if substituting t for all free occurrences of x in φ does not introduce a new bound occurrence of a variable from t.
- Then we denote the obtained formula $\varphi(x/t)$ and we call it an *instance* of the formula φ after a *substitution* of a term t for a variable x.
- t is not substitutable for x in φ if and only if x has a free occurrence in some subformula that starts with $(\forall y)$ or $(\exists y)$ for some variable y in t.
- Ground terms are always substitutable.

Variants

Quantified variables can be (under certain conditions) renamed so that we obtain an equivalent formula.

Let $(Qx)\psi$ be a subformula of φ where Q means $\forall$ or $\exists$ and y is a variable such that the following conditions hold.

1) y is substitutable for x in ψ, and
2) y does not have a free occurrence in ψ.

Then by replacing the subformula $(Qx)\psi$ with $(Qy)\psi(x/y)$ we obtain a *variant* of φ *in subformula* $(Qx)\psi$. After variation of one or more subformulas in φ we obtain a *variant* of φ. *For example,*

$(\exists x)(\forall y)(x \leq y)$	*is a formula* φ,
$(\exists u)(\forall v)(u \leq v)$	*is a variant of* φ,
$(\exists y)(\forall y)(y \leq y)$	*is not a variant of* φ, 1) *does not hold*,
$(\exists x)(\forall x)(x \leq x)$	*is not a variant of* φ, 2) *does not hold*.

Structures

- $\underline{S} = \langle S, \leq \rangle$ is an ordered set where $\leq$ is reflexive, antisymmetric, transitive binary relation on S,
- $G = \langle V, E \rangle$ is an undirected graph without loops where V is the set of *vertices* and E is irreflexive, symmetric binary relation on V *(adjacency)*,
- $\underline{\mathbb{Z}_p} = \langle \mathbb{Z}_p, +, -, 0 \rangle$ is the additive group of integers modulo p,
- $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ is the field of rational numbers,
- $\underline{\mathcal{P}(X)} = \langle \mathcal{P}(X), -, \cap, \cup, \emptyset, X \rangle$ is the set algebra over X,
- $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ is the standard model of arithmetic,
- finite automata and other models of computation,
- relational databases, ...

A structure for a language

Let $L = \langle \mathcal{R}, \mathcal{F} \rangle$ be a signature of a language and A be a nonempty set.

- A *realization* (*interpretation*) of a *relation symbol* $R \in \mathcal{R}$ on A is any relation $R^A \subseteq A^{\mathrm{ar}(R)}$. A realization of $=$ on A is the relation Id_A (identity).
- A *realization* (*interpretation*) of a *function symbol* $f \in \mathcal{F}$ on A is any function $f^A \colon A^{\mathrm{ar}(f)} \to A$. Thus a realization of a constant symbol is some element of A.

A *structure* for the language L (*L-structure*) is a triple $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$, where

- A is nonempty set, called the *domain* of the structure $\mathcal{A}$,
- $\mathcal{R}^A = \langle R^A \mid R \in \mathcal{R} \rangle$ is a collection of realizations of relation symbols,
- $\mathcal{F}^A = \langle f^A \mid f \in \mathcal{F} \rangle$ is a collection of realizations of function symbols.

A structure for the language L is also called a *model of the language* L. The class of all models of L is denoted by $M(L)$. *Examples for* $L = \langle \leq \rangle$ *are*

$$\langle \mathbb{N}, \leq \rangle,\ \langle \mathbb{Q}, > \rangle,\ \langle X, E \rangle,\ \langle \mathcal{P}(X), \subseteq \rangle.$$

Value of terms

Let t be a term of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure.

- A *variable assignment* over the domain A is a function $e\colon \mathrm{Var} \to A$.
- The *value* $t^A[e]$ of the term t in the structure $\mathcal{A}$ under the assignment e is defined by

$$x^A[e] = e(x) \quad \text{for every } x \in \mathrm{Var},$$
$$(f(t_1, \ldots, t_n))^A[e] = f^A(t_1^A[e], \ldots, t_n^A[e]) \quad \text{for every } f \in \mathcal{F}.$$

- In particular, for a constant symbol c we have $c^A[e] = c^A$.
- If t is a ground term, its value in $\mathcal{A}$ is independent on the assignment e.
- The value of t in $\mathcal{A}$ depends only on the assignment of variables in t.

For example, the value of the term $x + 1$ in the structure $\mathcal{N} = \langle \mathbb{N}, +, 1 \rangle$ under the assignment e with $e(x) = 2$ is $(x + 1)^N[e] = 3$.

Values of atomic formulas

Let φ be an atomic formula of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ in the form $R(t_1, \ldots, t_n)$, $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure, and e be a variable assignment over A.

- The *truth value* $\mathrm{TVal}^A(\varphi)[e]$ of φ in $\mathcal{A}$ under the assignment e is

  $$\mathrm{TVal}^A(R(t_1, \ldots, t_n))[e] = \begin{cases} 1 & \text{if } (t_1^A[e], \ldots, t_n^A[e]) \in R^A, \\ 0 & \text{otherwise.} \end{cases}$$

  where $=^A$ is Id_A; that is, $\mathrm{TVal}^A(t_1 = t_2)[e] = 1$ if $t_1^A[e] = t_2^A[e]$, and $\mathrm{TVal}^A(t_1 = t_2)[e] = 0$ otherwise.
- If φ is a sentence; that is, all its terms are ground, then its value in $\mathcal{A}$ is independent on the assignment e.
- The value of φ in $\mathcal{A}$ depends only on the assignment of variables in φ.

For example, the value of φ in form $x + 1 \leq 1$ in $\mathcal{N} = \langle \mathbb{N}, +, 1, \leq \rangle$ under the assignment e is $\mathrm{TVal}^N(\varphi)[e] = 1$ if and only if $e(x) = 0$.

Values of non-atomic formulas

The *truth value* $\mathrm{TVal}^{\mathcal{A}}(\varphi)[e]$ of non-atomic φ in $\mathcal{A}$ under e is given by

$$
\begin{aligned}
\mathrm{TVal}^{\mathcal{A}}(\neg\varphi)[e] &= -_1(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e])\\
\mathrm{TVal}^{\mathcal{A}}(\varphi\wedge\psi)[e] &= \wedge_1(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e], \mathrm{TVal}^{\mathcal{A}}(\psi)[e])\\
\mathrm{TVal}^{\mathcal{A}}(\varphi\vee\psi)[e] &= \vee_1(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e], \mathrm{TVal}^{\mathcal{A}}(\psi)[e])\\
\mathrm{TVal}^{\mathcal{A}}(\varphi\rightarrow\psi)[e] &= \rightarrow_1(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e], \mathrm{TVal}^{\mathcal{A}}(\psi)[e])\\
\mathrm{TVal}^{\mathcal{A}}(\varphi\leftrightarrow\psi)[e] &= \leftrightarrow_1(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e], \mathrm{TVal}^{\mathcal{A}}(\psi)[e])\\
\mathrm{TVal}^{\mathcal{A}}((\forall x)\varphi)[e] &= \min_{a\in A}(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e(x/a)])\\
\mathrm{TVal}^{\mathcal{A}}((\exists x)\varphi)[e] &= \max_{a\in A}(\mathrm{TVal}^{\mathcal{A}}(\varphi)[e(x/a)])
\end{aligned}
$$

where $-_1$, $\wedge_1$, $\vee_1$, $\rightarrow_1$, $\leftrightarrow_1$ are the Boolean functions given by the tables and $e(x/a)$ for $a\in A$ denotes the assignment obtained from e by setting $e(x)=a$.

Observation $\mathrm{TVal}^{\mathcal{A}}(\varphi)[e]$ *depends only on the assignment of free variables.*

Validity in a structure under an assignment

A formula φ is valid in a structure $\mathcal{A}$ under assignment e if $\mathrm{TVal}^{A}(\varphi)[e] = 1$. Then we write $\mathcal{A} \models \varphi[e]$, and $\mathcal{A} \not\models \varphi[e]$ otherwise. It holds that

$$
\begin{array}{lcl}
\mathcal{A} \models \neg\varphi[e] & \Leftrightarrow & \mathcal{A} \not\models \varphi[e] \\
\mathcal{A} \models (\varphi \wedge \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ and } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \vee \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ or } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \rightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ implies } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \leftrightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ if and only if } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\forall x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for every } a \in A \\
\mathcal{A} \models (\exists x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for some } a \in A
\end{array}
$$

Observation *Let term t be substitutable for x in φ and ψ be a variant of φ. Then for every structure $\mathcal{A}$ and assignment e*

1) $\mathcal{A} \models \varphi(x/t)[e]$ *if and only if* $\mathcal{A} \models \varphi[e(x/a)]$ *where* $a = t^{A}[e]$,
2) $\mathcal{A} \models \varphi[e]$ *if and only if* $\mathcal{A} \models \psi[e]$.

Validity in a structure

Let φ be a formula of a language L and $\mathcal{A}$ be an L-structure.

- φ is *valid* (*true*) *in the structure* $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi$, if $\mathcal{A} \models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$. If φ is not valid in $\mathcal{A}$, we write $\mathcal{A} \not\models \varphi$.
- φ is *contradictory in* $\mathcal{A}$ if $\mathcal{A} \models \neg\varphi$; that is, $\mathcal{A} \not\models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$.
- For every formulas φ, ψ, variable x, and structure $\mathcal{A}$

$$\begin{array}{llll}
(1) & \mathcal{A} \models \varphi & \Rightarrow & \mathcal{A} \not\models \neg\varphi \\
(2) & \mathcal{A} \models \varphi \wedge \psi & \Leftrightarrow & \mathcal{A} \models \varphi \text{ and } \mathcal{A} \models \psi \\
(3) & \mathcal{A} \models \varphi \vee \psi & \Leftarrow & \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi \\
(4) & \mathcal{A} \models \varphi & \Leftrightarrow & \mathcal{A} \models (\forall x)\varphi
\end{array}$$

- If φ is a sentence, it is valid or contradictory in $\mathcal{A}$, and thus (1) holds also in $\Leftarrow$. If moreover ψ is a sentence, also (3) holds in $\Rightarrow$.
- By (4), $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \psi$ where ψ is a *universal closure* of φ, i.e. a formula $(\forall x_1) \cdots (\forall x_n)\varphi$ where $x_1, \ldots, x_n$ are all free variables in φ.

Validity in a theory

- A *theory* of language L is any set T of formulas of L (so called *axioms*).
- A *model of a theory* T is an L-structure $\mathcal{A}$ such that $\mathcal{A} \models \varphi$ for every $\varphi \in T$. Then we write $\mathcal{A} \models T$ and we say that $\mathcal{A}$ *satisfies* T.
- The *class of models* of a theory T is $M(T) = \{\mathcal{A} \in M(L) \mid \mathcal{A} \models T\}$.
- A formula φ is *valid (true) in T*, denoted by $T \models \varphi$, if $\mathcal{A} \models \varphi$ for every model $\mathcal{A}$ of T. Otherwise, we write $T \not\models \varphi$.
- φ is *contradictory in T* if $T \models \neg\varphi$, i.e. φ is contradictory in all models of T.
- φ is *independent in T* if it is neither valid nor contradictory in T.
- If $T = \emptyset$, we have $M(T) = M(L)$ and we omit T, eventually we say *"in logic"*. Then $\models \varphi$ means that φ is (*universally*) *valid* (a *tautology*).
- A *consequence* of T is the set $\theta^L(T)$ of all sentences of L valid in T, i.e.

$$\theta^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \models \varphi \text{ and } \varphi \text{ is a sentence}\}.$$

Example of a theory

A *theory of orderings* T in language $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \qquad \text{(reflexivity)}$$

$$x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y \qquad \text{(antisymmetry)}$$

$$x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z \qquad \text{(transitivity)}$$

Models of T are L-structures $\langle S, \leq_S \rangle$, so called ordered sets, that satisfy the axioms of T, for example $\mathcal{A} = \langle \mathbb{N}, \leq \rangle$ or $\mathcal{B} = \langle \mathcal{P}(X), \subseteq \rangle$ for $X = \{0, 1, 2\}$.

- *A formula* $\varphi\colon x \leq y \vee y \leq x$ *is valid in* $\mathcal{A}$ *but not in* $\mathcal{B}$ *since* $\mathcal{B} \not\models \varphi[e]$ *for the assignment* $e(x) = \{0\}, e(y) = \{1\}$, *thus* φ *is independent in* T.
- *A sentence* $\psi\colon (\exists x)(\forall y)(y \leq x)$ *is valid in* $\mathcal{B}$ *and contradictory in* $\mathcal{A}$, *hence it is independent in* T *as well. We write* $\mathcal{B} \models \psi$, $\mathcal{A} \models \neg\psi$.
- *A formula* $\chi\colon (x \leq y \wedge y \leq z \wedge z \leq x) \rightarrow (x = y \wedge y = z)$ *is valid in* T, *denoted by* $T \models \chi$, *the same holds for its universal closure.*

Unsatisfiability and validity

The problem of validity in a theory can be transformed to the problem of satisfiability of (another) theory.

Proposition *For every theory T and sentence φ (of the same language)*

$$T, \neg\varphi \text{ is unsatisfiable} \quad \Leftrightarrow \quad T \models \varphi.$$

Proof By definitions, it is equivalent that

(1) $T, \neg\varphi$ is unsatisfiable (i.e. it has no model),

(2) $\neg\varphi$ is not valid in any model of T,

(3) φ is valid in every model of T,

(4) $T \models \varphi$. $\square$

Remark *The assumption that φ is a sentence is necessary for $(2) \Rightarrow (3)$.*

For example, the theory $\{P(c), \neg P(x)\}$ is unsatisfiable, but $P(c) \not\models P(x)$, where P is a unary relation symbol and c is a constant symbol.

Basic algebraic theories

- theory of *groups* in the language $L = \langle +, -, 0 \rangle$ with equality has axioms

$$x + (y + z) = (x + y) + z \qquad \text{(associativity of } +\text{)}$$
$$0 + x = x = x + 0 \qquad \text{(} 0 \text{ is neutral to } +\text{)}$$
$$x + (-x) = 0 = (-x) + x \qquad \text{(} -x \text{ is inverse of } x\text{)}$$

- theory of *Abelian groups* has moreover ax. $x + y = y + x$ (commutativity)
- theory of *rings* in $L = \langle +, -, \cdot, 0, 1 \rangle$ with equality has moreover axioms

$$1 \cdot x = x = x \cdot 1 \qquad \text{(} 1 \text{ is neutral to } \cdot\text{)}$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad \text{(associativity of } \cdot\text{)}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z,\ (x + y) \cdot z = x \cdot z + y \cdot z \qquad \text{(distributivity)}$$

- theory of *commutative rings* has moreover ax. $x \cdot y = y \cdot x$ (commutativity)
- theory of *fields* in the same language has additional axioms

$$x \neq 0 \rightarrow (\exists y)(x \cdot y = 1) \qquad \text{(existence of inverses to } \cdot\text{)}$$
$$0 \neq 1 \qquad \text{(nontriviality)}$$

Properties of theories

A theory T of a language L is *(semantically)*

- *inconsistent* if $T \models \perp$, otherwise T is *consistent* (*satisfiable*),
- *complete* if it is consistent and every sentence of L is valid in T or contradictory in T,
- an *extension* of a theory T' of language L' if $L' \subseteq L$ and $\theta^{L'}(T') \subseteq \theta^{L}(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\theta^{L'}(T') = \theta^{L}(T) \cap \mathrm{Fm}_{L'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa,

Structures $\mathcal{A}$, $\mathcal{B}$ for a language L are *elementarily equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same sentences of L.

Observation *Let T and T' be theories of a language L. T is (semantically)*

(1) *consistent if and only if it has a model,*

(2) *complete iff it has a single model, up to elementary equivalence,*

(3) *an extension of T' if and only if $M(T) \subseteq M(T')$,*

(4) *equivalent with T' if and only if $M(T) = M(T')$.*