# Propositional and Predicate Logic - VII

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# Substructures

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ and $\mathcal{B} = \langle B, \mathcal{R}^B, \mathcal{F}^B \rangle$ be structures for $L = \langle \mathcal{R}, \mathcal{F} \rangle$.

We say that $\mathcal{B}$ is an (induced) *substructure* of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if

$(i)$ $B \subseteq A$,

$(ii)$ $R^B = R^A \cap B^{\mathrm{ar}(R)}$ for every $R \in \mathcal{R}$,

$(iii)$ $f^B = f^A \cap (B^{\mathrm{ar}(f)} \times B)$; that is, $f^B = f^A \restriction B^{\mathrm{ar}(f)}$, for every $f \in \mathcal{F}$.

A set $C \subseteq A$ is a domain of some substructure of $\mathcal{A}$ if and only if $C$ is closed under all functions of $\mathcal{A}$. Then the respective substructure, denoted by $\mathcal{A} \restriction C$, is said to be the *restriction* of the structure $\mathcal{A}$ to $C$.

- A set $C \subseteq A$ is *closed* under a function $f\colon A^n \to A$ if $f(x_1, \ldots, x_n) \in C$ for every $x_1, \ldots, x_n \in C$.

*Example:* $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$ *is a substructure of* $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$ *and* $\underline{\mathbb{Z}} = \underline{\mathbb{Q}} \restriction \mathbb{Z}$. *Furthermore,* $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *is their substructure and* $\underline{\mathbb{N}} = \underline{\mathbb{Q}} \restriction \mathbb{N} = \underline{\mathbb{Z}} \restriction \mathbb{N}$.

# Validity in a substructure

Let $\mathcal{B}$ be a substructure of a structure $\mathcal{A}$ for a (fixed) language $L$.

**Proposition** *For every open formula $\varphi$ and assignment $e\colon \mathrm{Var} \to B$,*

$$\mathcal{A} \models \varphi[e] \quad \textit{if and only if} \quad \mathcal{B} \models \varphi[e].$$

*Proof* For atomic $\varphi$ it follows from the definition of the truth value with respect to an assignment. Otherwise by induction on the structure of the formula. □

**Corollary** *For every open formula $\varphi$ and structure $\mathcal{A}$,*

$$\mathcal{A} \models \varphi \quad \textit{if and only if} \quad \mathcal{B} \models \varphi \textit{ for every substructure } \mathcal{B} \subseteq \mathcal{A}.$$

- A theory $T$ is *open* if all axioms of $T$ are open.

**Corollary** *Every substructure of a model of an open theory $T$ is a model of $T$.*

*For example, every substructure of a graph, i.e. a model of theory of graphs, is a graph, called a subgraph. Similarly subgroups, Boolean subalgebras, etc.*

# Generated substructure, expansion, reduct

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be a structure and $X \subseteq A$. Let $B$ be the smallest subset of $A$ containing $X$ that is closed under all functions of the structure $\mathcal{A}$ (including constants). Then the structure $\mathcal{A} \upharpoonright B$ is denoted by $\mathcal{A}\langle X \rangle$ and is called the substructure of $\mathcal{A}$ *generated* by the set $X$.

*Example: for* $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$, $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$, $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *it is* $\underline{\mathbb{Q}}\langle \{1\} \rangle = \underline{\mathbb{N}}$, $\underline{\mathbb{Q}}\langle \{-1\} \rangle = \underline{\mathbb{Z}}$, *and* $\underline{\mathbb{Q}}\langle \{2\} \rangle$ *is the substructure on all even natural numbers.*

Let $\mathcal{A}$ be a structure for a language $L$ and $L' \subseteq L$. By omitting realizations of symbols that are not in $L'$ we obtain from $\mathcal{A}$ a structure $\mathcal{A}'$ called the *reduct* of $\mathcal{A}$ to the language $L'$. Conversely, $\mathcal{A}$ is an *expansion* of $\mathcal{A}'$ into $L$.

*For example,* $\langle \mathbb{N}, + \rangle$ *is a reduct of* $\langle \mathbb{N}, +, \cdot, 0 \rangle$*. On the other hand, the structure* $\langle \mathbb{N}, +, c_i \rangle_{i \in \mathbb{N}}$ *with* $c_i = i$ *for every* $i \in \mathbb{N}$ *is the expansion of* $\langle \mathbb{N}, + \rangle$ *by names of elements from* $\mathbb{N}$*.*

# Theorem on constants

**Theorem** *Let $\varphi$ be a formula in a language $L$ with free variables $x_1, \ldots, x_n$ and let $T$ be a theory in $L$. Let $L'$ be the extension of $L$ with new constant symbols $c_1, \ldots, c_n$ and let $T'$ denote the theory $T$ in $L'$. Then*

$$T \models \varphi \quad \text{if and only if} \quad T' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

*Proof* ($\Rightarrow$) If $\mathcal{A}'$ is a model of $T'$, let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to $L$. Since $\mathcal{A} \models \varphi[e]$ for every assignment $e$, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

($\Leftarrow$) If $\mathcal{A}$ is a model of $T$ and $e$ an assignment, let $\mathcal{A}'$ be the expansion of $A$ into $L'$ by setting $c_i^{A'} = e(x_i)$ for every $i$. Since $\mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n)[e']$ for every assignment $e'$, we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A} \models \varphi[e]. \quad \square$$

# Extensions of theories

**Proposition** *Let $T$ be a theory of $L$ and $T'$ be a theory of $L'$ where $L \subseteq L'$.*

$(i)$ *$T'$ is an extension of $T$ if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of $T'$ to the language $L$ is a model of $T$,*

$(ii)$ *$T'$ is a conservative extension of $T$ if $T'$ is an extension of $T$ and every model $\mathcal{A}$ of $T$ can be expanded to the language $L'$ on a model $\mathcal{A}'$ of $T'$.*

*Proof*

$(i)a)$ If $T'$ is an extension of $T$ and $\varphi$ is any axiom of $T$, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of $T$.

$(i)b)$ If $\mathcal{A}$ is a model of $T$ and $T \models \varphi$ where $\varphi$ is of $L$, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus $T'$ is an extension of $T$.

$(ii)$ If $T' \models \varphi$ where $\varphi$ is of $L$ and $\mathcal{A}$ is a model of $T$, then in its expansion $\mathcal{A}'$ that models $T'$ we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore $T'$ is conservative. $\square$

# Extensions by definition of a relation symbol

Let $T$ be a theory of $L$, $\psi(x_1, \ldots, x_n)$ be a formula of $L$ in free variables $x_1, \ldots, x_n$ and $L'$ denote the language $L$ with a new $n$-ary relation symbol $R$.

The *extension* of $T$ *by definition of $R$* with the formula $\psi$ is the theory $T'$ of $L'$ obtained from $T$ by adding the axiom

$$R(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)$$

**Observation** *Every model of $T$ can be uniquely expanded to a model of $T'$.*

**Corollary** *$T'$ is a conservative extension of $T$.*

**Proposition** *For every formula $\varphi'$ of $L'$ there is $\varphi$ of $L$ s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

*Proof* Replace each subformula $R(t_1, \ldots, t_n)$ in $\varphi$ with $\psi'(x_1/t_1, \ldots, x_n/t_n)$, where $\psi'$ is a suitable variant of $\psi$ allowing all substitutions. $\square$

*For example, the symbol $\leq$ can be defined in arithmetics by the axiom*

$$x \leq y \leftrightarrow (\exists z)(x + z = y)$$

# Extensions by definition of a function symbol

Let $T$ be a theory of a language $L$ and $\psi(x_1, \ldots, x_n, y)$ be a formula of $L$ in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \ \wedge \ \psi(x_1, \ldots, x_n, z) \ \rightarrow \ y = z \qquad \text{(uniqueness)}$$

Let $L'$ denote the language $L$ with a new $n$-ary function symbol $f$.

The *extension* of $T$ *by definition of f* with the formula $\psi$ is the theory $T'$ of $L'$ obtained from $T$ by adding the axiom

$$f(x_1, \ldots, x_n) = y \ \leftrightarrow \ \psi(x_1, \ldots, x_n, y)$$

*Remark* *In particular, if $\psi$ is $t(x_1, \ldots, x_n) = y$ where $t$ is a term and $x_1, \ldots, x_n$ are the variables in $t$, both the conditions of existence and uniqueness hold.*

*For example binary $-$ can be defined using $+$ and unary $-$ by the axiom*

$$x - y = z \ \leftrightarrow \ x + (-y) = z$$

# Extensions by definition of a function symbol (cont.)

**Observation** *Every model of $T$ can be uniquely expanded to a model of $T'$.*

**Corollary** *$T'$ is a conservative extension of $T$.*

**Proposition** *For every formula $\varphi'$ of $L'$ there is $\varphi$ of $L$ s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

*Proof* It suffices to consider $\varphi'$ with a single occurrence of $f$. If $\varphi'$ has more, we may proceed inductively. Let $\varphi^*$ denote the formula obtained from $\varphi'$ by replacing the term $f(t_1, \ldots, t_n)$ with a new variable $z$. Let $\varphi$ be the formula

$$(\exists z)(\varphi^* \ \wedge \ \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where $\psi'$ is a suitable variant of $\psi$ allowing all substitutions.

Let $\mathcal{A}$ be a model of $T'$, $e$ be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment $e$, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. $\square$

# Extensions by definitions

A theory $T'$ of $L'$ is called an *extension* of a theory $T$ of $L$ *by definitions* if it is obtained from $T$ by successive definitions of relation and function symbols.

**Corollary** *Let $T'$ be an extension of a theory $T$ by definitions. Then*

- *every model of $T$ can be uniquely expanded to a model of $T'$,*
- *$T'$ is a conservative extension of $T$,*
- *for every formula $\varphi'$ of $L'$ there is a formula $\varphi$ of $L$ such that $T' \models \varphi' \leftrightarrow \varphi$.*

*For example, in $T = \{(\exists y)(x+y=0), (x+y=0) \wedge (x+z=0) \rightarrow y=z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms*

$$-x = y \quad \leftrightarrow \quad x+y=0$$
$$x < y \quad \leftrightarrow \quad x \leq y \ \wedge \ \neg(x=y)$$

*Then the formula $-x < y$ is equivalent in this extension to a formula*

$$(\exists z)((z \leq y \ \wedge \ \neg(z=y)) \ \wedge \ x+z=0).$$

# Definable sets

*We interested in which sets can be defined within a given structure.*

- *A set defined by a formula* $\varphi(x_1, \ldots, x_n)$ *in structure* $\mathcal{A}$ is the set
$$\varphi^{\mathcal{A}}(x_1, \ldots, x_n) = \{(a_1, \ldots, a_n) \in A^n \mid \mathcal{A} \models \varphi[e(x_1/a_1, \ldots, x_n/a_n)]\}.$$
Shortly, $\varphi^{\mathcal{A}}(\overline{x}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a})]\}$, where $|\overline{x}| = n$.
- *A set defined by a formula* $\varphi(\overline{x}, \overline{y})$ *with parameters* $\overline{b} \in A^{|\overline{y}|}$ in $\mathcal{A}$ is
$$\varphi^{\mathcal{A}, \overline{b}}(\overline{x}, \overline{y}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a}, \overline{y}/\overline{b})]\}.$$
*Example:* $E(x, y)^{\mathcal{G}, b}$ *is the set of neighbors of a vertex* $b$ *in a graph* $\mathcal{G}$.
- For a structure $\mathcal{A}$, a set $B \subseteq A$, and $n \in \mathbb{N}$ let $\mathrm{Df}^n(\mathcal{A}, B)$ denote the class of definable sets $D \subseteq A^n$ in the structure $\mathcal{A}$ with parameters from $B$.

**Observation** $\mathrm{Df}^n(\mathcal{A}, B)$ *is closed under complements, union, intersection and it contains* $\emptyset$, $A^n$. *Thus it forms a subalgebra of the set algebra* $\underline{\mathcal{P}}(A^n)$.

# Example - database queries

| *Movie* | *name* | *director* | *actor* |
|---|---|---|---|
| | Lidé z Maringotek | M. Frič | J. Tříska |
| | Po strništi bos | J. Svěrák | Z. Svěrák |
| | Po strništi bos | J. Svěrák | J. Tříska |
| | ... | ... | ... |

| *Program* | *cinema* | *name* | *time* |
|---|---|---|---|
| | Světozor | Po strništi bos | 13:15 |
| | Mat | Po strništi bos | 16:15 |
| | Mat | Lidé z Maringotek | 18:30 |
| | ... | ... | ... |

*Where and when can I see a movie with J. Tříska?*

**select** *Program.cinema, Program.time* **from** *Movie, Program*
**where** *Movie.name* = *Program.name* **and** *actor* = 'J. Tříska';

Equivalently, it is the set $\varphi^{\mathcal{D}}(x, y)$ defined by the formula $\varphi(x, y)$

$$(\exists n)(\exists d)(P(x, n, y) \wedge M(n, d, \text{'J. Tříska'}))$$

in the structure $\mathcal{D} = \langle D, Movie, Program, c^D\rangle_{c\in D}$ of $L = \langle M, P, c\rangle_{c\in D}$, where $D = \{$'Po strništi bos', 'J. Tříska', 'Mat', '13:15', $\ldots\}$ and $c^D = c$ for any $c \in D$.

# Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1\rangle$ with equality and the following axioms.

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity of $\wedge$)

$x \vee (y \vee z) = (x \vee y) \vee z$ (associativity of $\vee$)

$x \wedge y = y \wedge x$ (commutativity of $\wedge$)

$x \vee y = y \vee x$ (commutativity of $\vee$)

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity of $\wedge$ over $\vee$)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (distributivity of $\vee$ over $\wedge$)

$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$ (absorption)

$x \vee (-x) = 1, \quad x \wedge (-x) = 0$ (complementation)

$0 \neq 1$ (non-triviality)

The smallest model is $\underline{2} = \langle \{0,1\}, -_1, \wedge_1, \vee_1, 0, 1\rangle$. Finite Boolean algebras are (up to isomorphism) $\langle \{0,1\}^n, -_n, \wedge_n, \vee_n, 0_n, 1_n\rangle$ for $n \in \mathbb{N}^+$, where the operations *(on binary $n$-tuples)* are the coordinate-wise operations of $\underline{2}$.

# Relations of propositional and predicate logic

- Propositional formulas over connectives $\neg$, $\wedge$, $\vee$ (eventually with $\top$, $\bot$) can be viewed as Boolean terms. Then the truth value of $\varphi$ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- Algebra of propositions over $\mathbb{P}$ is a Boolean algebra (also for $\mathbb{P}$ infinite).
- If we represent atomic subformulas in an open formula $\varphi$ (without equality) with propositional letters, we obtain a proposition that is valid if and only if $\varphi$ is valid.
- Propositional logic can be introduced as a fragment of predicate logic using nullary relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.