Propositional and Predicate Logic - IX

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The canonical model

From a noncontradictory branch V of a finished tableau we build a model that agrees with V. We build it on available (syntactical) objects - ground terms.

Let V be a noncontradictory branch of a finished tableau from a theory T of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. The *canonical model* from V is the L_C-structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ where

(1) A is the set of all ground terms of the language L_C,

(2) $f^A(t_{i_1}, \ldots, t_{i_n}) = f(t_{i_1}, \ldots, t_{i_n})$
for every n-ary function symbol $f \in \mathcal{F} \cup (L_C \setminus L)$ and $t_{i_1}, \ldots, t_{i_n} \in A$.

(3) $R^A(t_{i_1}, \ldots, t_{i_n}) \Leftrightarrow T\, R(t_{i_1}, \ldots, t_{i_n})$ is an entry on V
for every n-ary relation symbol $R \in \mathcal{R}$ or equality and $t_{i_1}, \ldots, t_{i_n} \in A$.

Remark *The expression $f(t_{i_1}, \ldots, t_{i_n})$ on the right side of (2) is a ground term of L_C, i.e. an element of A. Informally, to indicate that it is a syntactical object*

$$f^A(t_{i_1}, \ldots, t_{i_n}) = \text{“}f(t_{i_1}, \ldots, t_{i_n})\text{”}$$

The canonical model - an example

Let $T = \{(\forall x)R(f(x))\}$ be a theory of a language $L = \langle R, f, d\rangle$. The systematic tableau for $F\neg R(d)$ from T contains a single branch V, which is noncontradictory.

The canonical model $\mathcal{A} = \langle A, R^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V is for language L_C and

$$A = \{d, f(d), f(f(d)), \dots, c_0, f(c_0), f(f(c_0)), \dots, c_1, f(c_1), f(f(c_1)), \dots\},$$

$$d^A = d, \quad c_i^A = c_i \text{ for } i \in \mathbb{N},$$

$$f^A(d) = \text{“}f(d)\text{”}, \; f^A(f(d)) = \text{“}f(f(d))\text{”}, \; f^A(f(f(d))) = \text{“}f(f(f(d)))\text{”}, \; \dots$$

$$R^A = \{d, f(d), f(f(d)), \dots, f(c_0), f(f(c_0)), \dots, f(c_1), f(f(c_1)), \dots\}.$$

The reduct of $\mathcal{A}$ to the language L is $\mathcal{A}' = \langle A, R^A, f^A, d^A\rangle$.

The canonical model with equality

If L is with equality, T^* is the extension of T by the axioms of equality for L.

If we require that the equality is interpreted as the identity, we have to take the quotient of the canonical model $\mathcal{A}$ by the congruence $=^A$.

By (3), for the relation $=^A$ in $\mathcal{A}$ from V it holds that for every $s, t \in A$,

$$s =^A t \quad \Leftrightarrow \quad T(s = t) \text{ is an entry on } V.$$

Since V is finished and contains the axioms of equality, the relation $=^A$ is a congruence for all functions and relations in $\mathcal{A}$.

The *canonical model with equality* from V is the quotient $\mathcal{A}/{=^A}$.

Observation *For every formula φ,*

$$\mathcal{A} \models \varphi \quad \Leftrightarrow \quad (\mathcal{A}/{=^A}) \models \varphi,$$

where $=$ is interpreted in $\mathcal{A}$ by the relation $=^A$, while in $\mathcal{A}/{=^A}$ by the identity.

Remark *$\mathcal{A}$ is a countably infinite model, but $\mathcal{A}/{=^A}$ can be finite.*

The canonical model with equality - an example

Let $T = \{(\forall x)R(f(x)),\ (\forall x)(x = f(f(x)))\}$ be of $L = \langle R, f, d\rangle$ with equality.
The systematic tableau for $F\neg R(d)$ from T^* contains a noncontradictory V.

In the canonical model $\mathcal{A} = \langle A, R^A, =^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V we have that

$$s =^A t \quad\Leftrightarrow\quad t = f(\cdots(f(s)\cdots) \text{ or } s = f(\cdots(f(t)\cdots),$$

where f is applied $2i$-times for some $i \in \mathbb{N}$.

The canonical model with equality from V is
$\mathcal{B} = (\mathcal{A}/=^A) = \langle A/=^A, R^B, f^B, d^B, c_i^B\rangle_{i\in\mathbb{N}}$ where

$$\begin{aligned}
(A/=^A) &= \{[d]_{=^A}, [f(d)]_{=^A}, [c_0]_{=^A}, [f(c_0)]_{=^A}, [c_1]_{=^A}, [f(c_1)]_{=^A}, \ldots\},\\
d^B &= [d]_{=^A}, \quad c_i^B = [c_i]_{=^A} \text{ for } i \in \mathbb{N},\\
f^B([d]_{=^A}) &= [f(d)]_{=^A},\ f^B([f(d)]_{=^A}) = [f(f(d))]_{=^A} = [d]_{=^A},\ \ldots\\
R^B &= (A/=^A).
\end{aligned}$$

The reduct of $\mathcal{B}$ to the language L is $\mathcal{B}' = \langle A/=^A, R^B, f^B, d^B\rangle$.

Completeness

Lemma *The canonical model $\mathcal{A}$ from a noncontr. finished V agrees with V.*

Proof By induction on the structure of a sentence in an entry on V.

- For atomic φ, if $T\varphi$ is on V, then $\mathcal{A} \models \varphi$ by (3). If $F\varphi$ is on V, then $T\varphi$ is not on V since V is noncontradictory, so $\mathcal{A} \models \neg\varphi$ by (3).
- If $T(\varphi \wedge \psi)$ is on V, then $T\varphi$ and $T\psi$ are on V since V is finished. By induction, $\mathcal{A} \models \varphi$ and $\mathcal{A} \models \psi$, and thus $\mathcal{A} \models \varphi \wedge \psi$.
- If $F(\varphi \wedge \psi)$ is on V, then $F\varphi$ or $F\psi$ is on V since V is finished. By induction, $\mathcal{A} \models \neg\varphi$ or $\mathcal{A} \models \neg\psi$, and thus $\mathcal{A} \models \neg(\varphi \wedge \psi)$.
- For other connectives similarly as in previous two cases.
- If $T(\forall x)\varphi(x)$ is on V, then $T\varphi(x/t)$ is on V for every $t \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/t)$ for every $t \in A$, and thus $\mathcal{A} \models (\forall x)\varphi(x)$. Similarly for $F(\exists x)\varphi(x)$ on V.
- If $T(\exists x)\varphi(x)$ is on V, then $T\varphi(x/c)$ is on V for some $c \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/c)$, and thus $\mathcal{A} \models (\exists x)\varphi(x)$. Similarly for $F(\forall x)\varphi(x)$ on V. $\square$

Theorem on completeness

We will show that the tableau method in predicate logic is complete.

Theorem *For every theory T and sentence φ, if φ is valid in T, then φ is tableau provable from T, i.e. $T \models \varphi \;\Rightarrow\; T \vdash \varphi$.*

Proof Let φ be valid in T. We will show that an arbitrary finished tableau (e.g. systematic) τ from a theory T with the root entry $F\varphi$ is contradictory.

- If not, then there is some noncontradictory branch V in τ.
- By the previous lemma, there is a structure $\mathcal{A}$ for L_C that agrees with V, in particular with the root entry $F\varphi$, i.e. $\mathcal{A} \models \neg\varphi$.
- Let $\mathcal{A}'$ be the reduct of $\mathcal{A}$ to the language L. Then $\mathcal{A}' \models \neg\varphi$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus $\mathcal{A}'$ is a model of T (as $\mathcal{A}'$ agrees with $T\psi$ for every $\psi \in T$).
- But this contradicts the assumption that φ is valid in T.

Therefore the tableau τ is a proof of φ from T. $\square$

Properties of theories

We introduce syntactic variants of previous semantical definitions.

Let T be a theory of a language L. If a sentence φ is provable from T, we say that φ is a *theorem* of T. The set of theorems of T is denoted by

$$\mathrm{Thm}^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \bot$, otherwise T is *consistent*,
- *complete* if it is consistent and every sentence is provable or refutable from T, i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- an *extension* of a theory T' of L' if $L' \subseteq L$ and $\mathrm{Thm}^{L'}(T') \subseteq \mathrm{Thm}^L(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\mathrm{Thm}^{L'}(T') = \mathrm{Thm}^L(T) \cap \mathrm{Fm}_{\mathrm{L}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary *For every theory T and sentences φ, ψ of a language L,*

- $T \vdash \varphi$ *if and only if* $T \models \varphi$,
- $\mathrm{Thm}^L(T) = \theta^L(T)$,
- T *is inconsistent if and only if* T *is unsatisfiable, i.e. it has no model,*
- T *is complete if and only if* T *is semantically complete, i.e. it has a single model, up to elementary equivalence,*

Existence of a countable model and compactness

Theorem *Every consistent theory T of a countable language L without equality has a countably infinite model.*

Proof Let τ be the systematic tableau from T with $F\bot$ in the root. Since τ is finished and contains a noncontradictory branch V as $\bot$ is not provable from T, there exists a canonical model $\mathcal{A}$ from V. Since $\mathcal{A}$ agrees with V, its reduct to the language L is a desired countably infinite model of T. $\square$

Remark This is a weak version of so called Löwenheim-Skolem theorem. In a countable language with equality the canonical model with equality is countable (i.e. finite or countably infinite).

Theorem *A theory T has a model iff every finite subset of T has a model.*

Proof The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. $\bot$ is provable by a systematic tableau τ from T. Since τ is finite, $\bot$ is provable from some finite $T' \subseteq T$, i.e. T' has no model. $\square$

Non-standard model of natural numbers

Let $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ be the standard model of natural numbers.

Let $\mathrm{Th}(\underline{\mathbb{N}})$ denote the set of all sentences that are valid in $\underline{\mathbb{N}}$. For $n \in \mathbb{N}$ let $\underline{n}$ denote the term $S(S(\cdots(S(0))\cdots))$, so called the *n-th numeral*, where S is applied n-times.

Consider the following theory T where c is a new constant symbol.

$$T = \mathrm{Th}(\underline{\mathbb{N}}) \cup \{\underline{n} < c \mid n \in \mathbb{N}\}$$

Observation *Every finite subset of T has a model.*

Thus by the compactness theorem, T has a model $\mathcal{A}$. It is a non-standard model of natural numbers. Every sentence from $\mathrm{Th}(\underline{\mathbb{N}})$ is valid in $\mathcal{A}$ but it contains an element c^A that is greater then every $n \in \mathbb{N}$ (i.e. the value of the term $\underline{n}$ in $\mathcal{A}$).

Equisatisfiability

We will see that the problem of satisfiability can be reduced to open theories.

- Theories T, T' are *equisatisfiable* if T has a model $\Leftrightarrow$ T' has a model.
- A formula φ is in the *prenex (normal) form (PNF)* if it is written as
$$(Q_1 x_1) \dots (Q_n x_n) \varphi',$$
where Q_i denotes $\forall$ or $\exists$, variables $x_1, \dots, x_n$ are all distinct and φ' is an open formula, called the *matrix*. $(Q_1 x_1) \dots (Q_n x_n)$ is called the *prefix*.
- In particular, if all quantifiers are $\forall$, then φ is a *universal* formula.

To find an open theory equisatisfiable with T we proceed as follows.

(1) We replace axioms of T by equivalent formulas in the prenex form.

(2) We transform them, using new function symbols, to equisatisfiable universal formulas, so called Skolem variants.

(3) We take their matrices as axioms of a new theory.

Conversion rules for quantifiers

Let Q denote $\forall$ or $\exists$ and let $\overline{Q}$ denote the complementary quantifier. For every formulas φ, ψ such that x is not free in the formula ψ,

$$\begin{array}{lrcl}
\models & \neg(Qx)\varphi & \leftrightarrow & (\overline{Q}x)\neg\varphi \\
\models & ((Qx)\varphi \wedge \psi) & \leftrightarrow & (Qx)(\varphi \wedge \psi) \\
\models & ((Qx)\varphi \vee \psi) & \leftrightarrow & (Qx)(\varphi \vee \psi) \\
\models & ((Qx)\varphi \rightarrow \psi) & \leftrightarrow & (\overline{Q}x)(\varphi \rightarrow \psi) \\
\models & (\psi \rightarrow (Qx)\varphi) & \leftrightarrow & (Qx)(\psi \rightarrow \varphi)
\end{array}$$

The above equivalences can be verified semantically or proved by the tableau method (*by taking the universal closure if it is not a sentence*).

Remark *The assumption that x is not free in ψ is necessary in each rule above (except the first one) for some quantifier Q. For example,*

$$\not\models \; ((\exists x)P(x) \wedge P(x)) \; \leftrightarrow \; (\exists x)(P(x) \wedge P(x))$$

Conversion to the prenex normal form

Proposition *Let φ' be the formula obtained from φ by replacing some occurrences of a subformula ψ with ψ'. If $T \models \psi \leftrightarrow \psi'$, then $T \models \varphi \leftrightarrow \varphi'$.*

Proof Easily by induction on the structure of the formula φ. □

Proposition *For every formula φ there is an equivalent formula φ' in the prenex normal form, i.e. $\models \varphi \leftrightarrow \varphi'$.*

Proof By induction on the structure of φ applying the conversion rules for quantifiers, replacing subformulas with their variants if needed, and applying the above proposition on equivalent transformations. □

For example,

$$
\begin{aligned}
((\forall z)P(x,z) \wedge P(y,z)) &\rightarrow \neg(\exists x)P(x,y)\\
((\forall u)P(x,u) \wedge P(y,z)) &\rightarrow (\forall x)\neg P(x,y)\\
(\forall u)(P(x,u) \wedge P(y,z)) &\rightarrow (\forall v)\neg P(v,y)\\
(\exists u)((P(x,u) \wedge P(y,z)) &\rightarrow (\forall v)\neg P(v,y))\\
(\exists u)(\forall v)((P(x,u) \wedge P(y,z)) &\rightarrow \neg P(v,y))
\end{aligned}
$$

Skolem variants

Let φ be a sentence of a language L in the prenex normal form, let $y_1, \dots, y_n$ be the existentially quantified variables in φ (in this order), and for every $i \leq n$ let $x_1, \dots, x_{n_i}$ be the variables that are universally quantified in φ before y_i. Let L' be an extension of L with new n_i-ary function symbols f_i for all $i \leq n$.

Let φ_S denote the formula of L' obtained from φ by removing all $(\exists y_i)$'s from the prefix and by replacing each occurrence of y_i with the term $f_i(x_1, \dots, x_{n_i})$. Then φ_S is called a *Skolem variant* of φ.

For example, for the formula φ

$$(\exists y_1)(\forall x_1)(\forall x_2)(\exists y_2)(\forall x_3)R(y_1, x_1, x_2, y_2, x_3)$$

the following formula φ_S *is a Skolem variant of* φ

$$(\forall x_1)(\forall x_2)(\forall x_3)R(f_1, x_1, x_2, f_2(x_1, x_2), x_3),$$

where f_1 *is a new constant symbol and* f_2 *is a new binary function symbol.*

Properties of Skolem variants

Lemma *Let φ be a sentence $(\forall x_1)\dots(\forall x_n)(\exists y)\psi$ of L and φ' be a sentence $(\forall x_1)\dots(\forall x_n)\psi(y/f(x_1,\dots,x_n))$ where f is a new function symbol. Then*

(1) *the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of φ' to the language L is a model of φ,*

(2) *every model $\mathcal{A}$ of φ can be expanded into a model $\mathcal{A}'$ of φ'.*

Remark *Compared to extensions by definition of a function symbol, the expansion in (2) does not need to be unique now.*

Proof (1) Let $\mathcal{A}' \models \varphi'$ and $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \psi[e(y/a)]$ for every assignment e where $a = (f(x_1,\dots,x_n))^{A'}[e]$, we have also $\mathcal{A} \models \varphi$.
(2) Let $\mathcal{A} \models \varphi$. There exists a function $f^A\colon A^n \to A$ such that for every assignment e it holds $\mathcal{A} \models \psi[e(y/a)]$ where $a = f^A(e(x_1),\dots,e(x_n))$, and thus the expansion $\mathcal{A}'$ of $\mathcal{A}$ by the function f^A is a model of φ'. $\square$

Corollary *If φ' is a Skolem variant of φ, then both statements (1) and (2) hold for φ, φ' as well. Hence φ, φ' are equisatisfiable.*

Skolem's theorem

Theorem *Every theory T has an open conservative extension T^*.*

Proof We may assume that T is in a closed form. Let L be its language.

- By replacing each axiom of T with an equivalent formula in the prenex normal form we obtain an equivalent theory T°.
- By replacing each axiom of T° with its Skolem variant we obtain a theory T' in an extended language $L' \supseteq L$.
- Since the reduct of every model of T' to the language L is a model of T, the theory T' is an extension of T.
- Furthermore, since every model of T can be expanded to a model of T', it is a conservative extension.
- Since every axiom of T' is a universal sentence, by replacing them with their matrices we obtain an open theory T^* equivalent to T'. $\square$

Corollary *For every theory there is an equisatisfiable open theory.*