Propositional and Predicate Logic - XI

Petr Gregor

KTIML MFF UK

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Theories of structures

What holds in particular structures?

The *theory of a structure* $\mathcal{A}$ is the set $\mathrm{Th}(\mathcal{A})$ of all sentences (of the same language) that are valid in $\mathcal{A}$.

Observation *For every structure* $\mathcal{A}$ *and a theory* T *of a language* L*,*

(i) $\mathrm{Th}(\mathcal{A})$ *is a complete theory,*

(ii) *if* $\mathcal{A} \models T$*, then* $\mathrm{Th}(\mathcal{A})$ *is a simple (complete) extension of* T*,*

(iii) *if* $\mathcal{A} \models T$ *and* T *is complete, then* $\mathrm{Th}(\mathcal{A})$ *is equivalent with* T*, i.e.* $\theta^L(T) = \mathrm{Th}(\mathcal{A})$*.*

E.g. $\mathrm{Th}(\underline{\mathbb{N}})$ *where* $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ *is the arithmetics of natural numbers.*

Remark *Later, we will see that* $\mathrm{Th}(\underline{\mathbb{N}})$ *is (algorithmically) undecidable although it is complete.*

Elementary equivalence

- Structures $\mathcal{A}$ and $\mathcal{B}$ of a language L are *elementarily equivalent*, denoted $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same sentences (of L), i.e. $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$.

  For example, $\langle \mathbb{R}, \leq \rangle \equiv \langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{Q}, \leq \rangle \not\equiv \langle \mathbb{Z}, \leq \rangle$ since every element has an immediate successor in $\langle \mathbb{Z}, \leq \rangle$ but not in $\langle \mathbb{Q}, \leq \rangle$.

- T is complete iff it has a single model, up to elementary equivalence.

  For example, the theory of dense linear orders without ends (DeLO).

How to describe models of a given theory (up to elementary equivalence)?

Observation *For every models $\mathcal{A}$, $\mathcal{B}$ of a theory T, $\mathcal{A} \equiv \mathcal{B}$ if and only if $\mathrm{Th}(\mathcal{A})$, $\mathrm{Th}(\mathcal{B})$ are equivalent (simple complete extensions of T).*

Remark *If we can describe effectively (recursively) for a given theory T all simple complete extensions of T, then T is (algorithmically) decidable.*

Simple complete extensions - an example

The theory $DeLO^*$ of dense linear orders of $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \qquad \text{(reflexivity)}$$
$$x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y \qquad \text{(antisymmetry)}$$
$$x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z \qquad \text{(transitivity)}$$
$$x \leq y \ \vee \ y \leq x \qquad \text{(dichotomy)}$$
$$x < y \ \rightarrow \ (\exists z)\,(x < z \ \wedge \ z < y) \qquad \text{(density)}$$
$$(\exists x)(\exists y)(x \neq y) \qquad \text{(nontriviality)}$$

where '$x < y$' is a shortcut for '$x \leq y \ \wedge \ x \neq y$'.

Let φ, ψ be the sentences $(\exists x)(\forall y)(x \leq y)$, resp. $(\exists x)(\forall y)(y \leq x)$. We will see

$$DeLO = DeLO^* \cup \{\neg\varphi, \neg\psi\}, \qquad DeLO^{\pm} = DeLO^* \cup \{\varphi, \psi\},$$
$$DeLO^{+} = DeLO^* \cup \{\neg\varphi, \psi\}, \qquad DeLO^{-} = DeLO^* \cup \{\varphi, \neg\psi\}$$

are the all (nonequivalent) simple complete extensions of the theory $DeLO^*$.

Corollary of the Löwenheim-Skolem theorem

We already know the following theorem, by a canonical model (with equality).

Theorem *Let T be a consistent theory of a countable language L. If L is without equality, then T has a countably infinite model. If L is with equality, then T has a model that is countable (finite or countably infinite).*

Corollary *For every structure $\mathcal{A}$ of a countable language without equality there exists a countably infinite structure $\mathcal{B}$ with $\mathcal{A} \equiv \mathcal{B}$.*

Proof $\mathrm{Th}(\mathcal{A})$ is consistent since it has a model $\mathcal{A}$. By the previous theorem, it has a countably inf. model $\mathcal{B}$. Since $\mathrm{Th}(\mathcal{A})$ is complete, we have $\mathcal{A} \equiv \mathcal{B}$. □

Corollary *For every infinite structure $\mathcal{A}$ of a countable language with equality there exists a countably infinite structure $\mathcal{B}$ with $\mathcal{A} \equiv \mathcal{B}$.*

Proof Similarly as above. Since the sentence *"there is exactly n elements"* is false in $\mathcal{A}$ for all n and $\mathcal{A} \equiv \mathcal{B}$, it follows that B is infinite. □

A countable algebraically closed field

We say that a field $\mathcal{A}$ is *algebraically closed* if every polynomial (of nonzero degree) has a root in $\mathcal{A}$; that is, for every $n \geq 1$ we have

$$\mathcal{A} \models (\forall x_{n-1}) \dots (\forall x_0)(\exists y)(y^n + x_{n-1} \cdot y^{n-1} + \dots + x_1 \cdot y + x_0 = 0)$$

where y^k is a shortcut for the term $y \cdot y \cdot \ \cdots \ \cdot y$ ($\cdot$ applied $(k-1)$-times).

For example, the field $\underline{\mathbb{C}} = \langle \mathbb{C}, +, -, \cdot, 0, 1 \rangle$ is algebraically closed, whereas the fields $\underline{\mathbb{R}}$ and $\underline{\mathbb{Q}}$ are not (since the polynomial $x^2 + 1$ has no root in them).

Corollary *There exists a countable algebraically closed field.*

Proof By the previous corollary, there is a countable structure elementarily equivalent with the field $\underline{\mathbb{C}}$. Hence it is algebraically closed as well. □

Isomorphisms of structures

Let $\mathcal{A}$ and $\mathcal{B}$ be structures of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$.

- A bijection $h\colon A \to B$ is an *isomorphism* of structures $\mathcal{A}$ and $\mathcal{B}$ if both
  - (i) $h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$
    for every n-ary function symbol $f \in \mathcal{F}$ and every $a_1, \ldots, a_n \in A$,
  - (ii) $R^A(a_1, \ldots, a_n) \;\Leftrightarrow\; R^B(h(a_1), \ldots, h(a_n))$
    for every n-ary relation symbol $R \in \mathcal{R}$ and every $a_1, \ldots, a_n \in A$.
- $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* (via h), denoted $\mathcal{A} \simeq \mathcal{B}$ ($\mathcal{A} \simeq_h \mathcal{B}$), if there is an isomorphism h of $\mathcal{A}$ and $\mathcal{B}$. We also say that $\mathcal{A}$ is *isomorphic with* $\mathcal{B}$.
- An *automorphism* of a structure $\mathcal{A}$ is an isomorphism of $\mathcal{A}$ with $\mathcal{A}$.

For example, the power set algebra $\underline{\mathcal{P}(X)} = \langle \mathcal{P}(X), -, \cap, \cup, \emptyset, X \rangle$ *with* $|X| = n$ *is isomorphic to the Boolean algebra* $\langle \{0,1\}^n, -_n, \wedge_n, \vee_n, 0_n, 1_n \rangle$ *via* $h : A \mapsto \chi_A$ *where* χ_A *is the characteristic function of the set* $A \subseteq X$.

Isomorphisms and semantics

We will see that isomorphism preserves semantics.

Proposition *Let $\mathcal{A}$ and $\mathcal{B}$ be structures of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. A bijection $h\colon A \to B$ is an isomorphism of $\mathcal{A}$ and $\mathcal{B}$ if and only if both*

(i) $h(t^A[e]) = t^B[e \circ h]$ *for every term t and $e\colon \mathrm{Var} \to A$, and*

(ii) $\mathcal{A} \models \varphi[e] \;\Leftrightarrow\; \mathcal{B} \models \varphi[e \circ h]$ *for every formula φ and $e\colon \mathrm{Var} \to A$.*

Proof $(\Rightarrow)$ By induction on the structure of the term t, resp. the formula φ.
$(\Leftarrow)$ By applying (i) for each term $f(x_1, \dots, x_n)$ or (ii) for each atomic formula $R(x_1, \dots, x_n)$ and assigning $e(x_i) = a_i$ we verify that h is an isomorphism. □

Corollary *For every structures $\mathcal{A}$ and $\mathcal{B}$ of the same language,*

$$\mathcal{A} \simeq \mathcal{B} \quad\Rightarrow\quad \mathcal{A} \equiv \mathcal{B}.$$

Remark *The other implication $(\Leftarrow)$ does not hold in general. For example, $\langle \mathbb{Q}, \leq \rangle \equiv \langle \mathbb{R}, \leq \rangle$ but $\langle \mathbb{Q}, \leq \rangle \not\simeq \langle \mathbb{R}, \leq \rangle$ since $|\mathbb{Q}| = \omega$ and $|\mathbb{R}| = 2^\omega$.*

Finite models in language with equality

Proposition *For every finite structures $\mathcal{A}$, $\mathcal{B}$ of a language with equality,*

$$\mathcal{A} \equiv \mathcal{B} \quad \Rightarrow \quad \mathcal{A} \simeq \mathcal{B}.$$

Proof It holds $|A| = |B|$ since we can express *"there are exactly n elements"*.

- Let $\mathcal{A}'$ be expansion of $\mathcal{A}$ to $L' = L \cup \{c_a\}_{a \in A}$ by names of elements of A.
- We show that $\mathcal{B}$ has an expansion $\mathcal{B}'$ to L' such that $\mathcal{A}' \equiv \mathcal{B}'$. Then clearly $h\colon a \mapsto c_a^{B'}$ is an isomorphism of $\mathcal{A}'$ to $\mathcal{B}'$, and thus also of $\mathcal{A}$ to $\mathcal{B}$.
- If suffices to find $b \in B$ for every $c_a^{A'} = a \in A$ such that $\langle \mathcal{A}, a \rangle \equiv \langle \mathcal{B}, b \rangle$.
- Let Ω be set of all formulas $\varphi(x)$ s.t. $\langle \mathcal{A}, a \rangle \models \varphi(x/c_a)$, i.e. $\mathcal{A} \models \varphi[e(x/a)]$
- Since A is finite, there are finitely many formulas $\varphi_0(x), \ldots, \varphi_m(x)$ such that for every $\varphi \in \Omega$ it holds $\mathcal{A} \models \varphi \leftrightarrow \varphi_i$ for some i.
- Since $\mathcal{B} \equiv \mathcal{A} \models (\exists x) \bigwedge_{i \leq m} \varphi_i$, there exists $b \in B$ s.t. $\mathcal{B} \models \bigwedge_{i \leq m} \varphi_i[e(x/b)]$.
- Hence for every $\varphi \in \Omega$ it holds $\mathcal{B} \models \varphi[e(x/b)]$, i.e. $\langle \mathcal{B}, b \rangle \models \varphi(x/c_a)$. $\square$

Corollary *If a complete theory T in a language with equality has a finite model, then all models of T are isomorphic.*

Definability and automorphisms

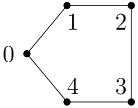
We show that definable sets are invariant under automorphisms.

Proposition *Let $D \subseteq A^n$ be a set definable in a structure $\mathcal{A}$ from parameters $\overline{b}$ and h be an automorphism of $\mathcal{A}$ that pointwise preserves $\overline{b}$. Then $h[D] = D$.*

Proof Let $D = \varphi^{\mathcal{A},\overline{b}}(\overline{x},\overline{y})$. Then for every $\overline{a} \in A^{|\overline{x}|}$

$$\overline{a} \in D \Leftrightarrow \mathcal{A} \models \varphi[e(\overline{x}/\overline{a},\overline{y}/\overline{b})] \Leftrightarrow \mathcal{A} \models \varphi[(e \circ h)(\overline{x}/\overline{a},\overline{y}/\overline{b})]$$
$$\Leftrightarrow \mathcal{A} \models \varphi[e(\overline{x}/h(\overline{a}),\overline{y}/h(\overline{b}))] \Leftrightarrow \mathcal{A} \models \varphi[e(\overline{x}/h(\overline{a}),\overline{y}/\overline{b})] \Leftrightarrow h(\overline{a}) \in D.$$

Ex.: the graph $\mathcal{G}$ has exactly one nontrivial automorphism h that preserves 0.



$h(0) = 0,\ h(1) = 4,\ h(2) = 3,\ h(3) = 2,\ h(4) = 1$

$\{0\} = (x = y)^{\mathcal{G},0}$, $\{1,4\} = (E(x,y))^{\mathcal{G},0}$, $\{2,3\} = (x \neq y \wedge \neg E(x,y))^{\mathcal{G},0}$

Moreover, the sets $\{0\}$, $\{1,4\}$, $\{2,3\}$ are definable with parameter 0. Thus
$\mathrm{Df}^1(\mathcal{G},\{0\}) = \{\emptyset,\{0\},\{1,4\},\{2,3\},\{0,1,4\},\{0,2,3\},\{1,4,2,3\},\{0,1,2,3,4\}\}.$

Categoricity

- An (isomorphism) *spectrum* of a theory T is given by the number $I(\kappa, T)$ of mutually nonisomorphic models of T for every cardinality κ.
- A theory T is *κ-categorical* if it has exactly one (up to isomorphism) model of cardinality κ, i.e. $I(\kappa, T) = 1$.

Proposition *The theory $DeLO$ (i.e. "without ends") is ω-categorical.*

Proof Let $\mathcal{A}, \mathcal{B} \models DeLO$ with $A = \{a_i\}_{i\in\mathbb{N}}$, $B = \{b_i\}_{i\in\mathbb{N}}$. By induction on n we can find injective partial functions $h_n \subseteq h_{n+1} \subset A \times B$ preserving the ordering s.t. $\{a_i\}_{i<n} \subseteq \mathrm{dom}(h_n)$ and $\{b_i\}_{i<n} \subseteq \mathrm{rng}(h_n)$. Then $\mathcal{A} \simeq \mathcal{B}$ via $h = \cup h_n$. □

Similarly we obtain that (e.g.) $\mathcal{A} = \langle \mathbb{Q}, \leq \rangle$, $\mathcal{A} \restriction (0, 1]$, $\mathcal{A} \restriction [0, 1)$, $\mathcal{A} \restriction [0, 1]$ are (up to isomorphism) all countable models of $DeLO^*$. Then

$$I(\kappa, DeLO^*) = \begin{cases} 0 & \text{for } \kappa \in \mathbb{N}, \\ 4 & \text{for } \kappa = \omega. \end{cases}$$

ω-categorical criterium of completeness

Theorem *Let L be at most countable language.*

(i) *If a theory T in L without equality is ω-categorical, then it is complete.*

(ii) *If a theory T in L with equality is ω-categorical and without finite models, then it is complete.*

Proof Every model of T is elementarily equivalent with some countably infinite model of T, but such model is unique up to isomorphism. Thus all models of T are elementarily equivalent, i.e. T is complete. $\square$

For example, DeLO, DeLO^+, DeLO^-, $\mathrm{DeLO}^\pm$ are complete and they are the all (mutually nonequivalent) simple complete extensions of DeLO^*.

Remark *A similar criterium holds also for cardinalities bigger than ω.*

Axiomatizability

We are interested if we can describe a class of models by given means.

Let $K \subseteq M(L)$ be a class of structures of a language L. We say that K is

- *axiomatizable* if there is a theory T of language L with $M(T) = K$,
- *finitely axiomatizable* if K is axiomatizable by a finite theory,
- *openly axiomatizable* if K is axiomatizable by an open theory,
- a theory T if finitely (openly) axiomatizable if T is equivalent to a finite (resp. open) theory.

Observation *If K is axiomatizable, then it is closed under elem. equivalence.*

For example,

$a)$ *linear orderings are both finitely and openly axiomatizable,*

$b)$ *fields are finitely axiomatizable, but not openly,*

$c)$ *infinite groups are axiomatizable, but not finitely.*

Application of compactness

Theorem *If a theory T has at least an n-element model for every $n \in \mathbb{N}$, then it also has an infinite model.*

Proof In a language without equality apply L.-S. theorem. Now assume we have a language with equality.

- Let $T' = T \cup \{c_i \neq c_j \mid \text{for } i \neq j\}$ be an extension of T in a language with additional countably infinitely many new constant symbols c_i.
- By the assumption, every finite part of T' has a model.
- By compactness, T' has a model, which clearly is infinite.
- Its reduct to the original language is an infinite model of T. □

Corollary *If a theory T has at least an n-element model for each $n \in \mathbb{N}$, the class of all its finite models is not axiomatizable.*

For example, finite groups, finite fields, etc. are not axiomatizable. But infinite models of a theory T in language with equality are axiomatizable.

Finite axiomatizability

Theorem *Let $K \subseteq M(L)$ and $\overline{K} = M(L) \setminus K$ where L is a language. Then K is finitely axiomatizable if and only if both K and $\overline{K}$ are axiomatizable.*

Proof $(\Rightarrow)$ If T is a finite axiomatization of K is a closed form, then the theory with the only axiom $\bigvee_{\varphi \in T} \neg\varphi$ axiomatizes $\overline{K}$. Now we show $(\Leftarrow)$.

- Let T, S be theories of language L such that $M(T) = K$, $M(S) = \overline{K}$.
- Then $M(T \cup S) = M(T) \cap M(S) = \emptyset$ and by the compactness there exist finite $T' \subseteq T$ and $S' \subseteq S$ such that $\emptyset = M(T' \cup S') = M(T') \cap M(S')$.
- Since
$$M(T) \subseteq M(T') \subseteq \overline{M(S')} \subseteq \overline{M(S)} = M(T),$$
we have $M(T) = M(T')$, i.e. a finite T' axiomatizes K. $\square$

Finite axiomatizability - example

Let T be the theory of fields. We say that a field $\mathcal{A} = \langle A, +, -, \cdot, 0, 1\rangle$ has

- *characteristic* 0 if there is no $p \in \mathbb{N}^+$ such that $\mathcal{A} \models p1 = 0$, where $p1$ denotes the term $1 + 1 + \cdots + 1$ ($+$ applied $(p-1)$-times).
- *characteristic* p where p is prime, if p is the smallest s.t. $\mathcal{A} \models p1 = 0$.
- The class of fields of characteristic p for prime p is finitely axiomatized by the theory $T \cup \{p1 = 0\}$.
- The class K of fields of characteristic 0 is axiomatized by the (infinite) theory $T' = T \cup \{p1 \neq 0 \mid p \in \mathbb{N}^+\}$.

Proposition *K is not finitely axiomatizable.*

Proof It suffices to show that $\overline{K}$ is not axiomatizable. Suppose $M(S) = \overline{K}$. Then $S' = S \cup T'$ has a model $\mathcal{B}$ since every finite $S^* \subseteq S'$ has a model (a field of prime characteristic larger than any p occurring in axioms of S^*), But then $\mathcal{B} \in M(S) = \overline{K}$ and $\mathcal{B} \in M(T') = K$, a contradiction. $\square$

Openly axiomatizable theories

Theorem *If a theory T is openly axiomatizable, then every substructure of a model of T is also a model of T.*

Proof Let T' be open axiomatization of $M(T)$, $\mathcal{A} \models T'$ and $\mathcal{B} \subseteq \mathcal{A}$. We know that $\mathcal{B} \models \varphi$ for every $\varphi \in T'$ since φ is open. Thus $\mathcal{B}$ is a model of T'. $\square$

Remark *The other implication holds as well, i.e. if every substructure of every model of T is also a model of T, then T is openly axiomatizable.*

For example, the theory DeLO is not openly axiomatizable since e.g. any finite substructure of a model of DeLO is not a model DeLO.

At most n-element groups for a fixed $n > 1$ are openly axiomatized by

$$T \cup \{ \bigvee_{\substack{i,j \leq n \\ i \neq j}} x_i = x_j \},$$

where T is the (open) theory of groups.