

Propositional and Predicate Logic - Appendix

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Set-theoretical notions

All notions are introduced within a **set theory** using only the membership predicate and equality (and means of logic).

- A property of sets $\varphi(x)$ defines a **class** $\{x \mid \varphi(x)\}$. A class that is not a set is called a **proper** class, eg. $\{x \mid x = x\}$,
- $x \notin y$, $x \neq y$ are shortcuts for $\neg(x \in y)$, $\neg(x = y)$,
- $\{x_0, \dots, x_{n-1}\}$ denotes the set containing exactly $x_0, \dots, x_{n-1}$, $\{x\}$ is called a **singleton**, $\{x, y\}$ is called an **unordered pair**,
- $\emptyset$, $\cup$, $\cap$, $\setminus$, Δ stand for **empty set**, **union**, **intersection**, **difference**, **symmetric difference** of sets, e.g.

$$x \Delta y = (x \setminus y) \cup (y \setminus x) = \{z \mid (z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y)\}$$

- x, y are **disjoint** if $x \cap y = \emptyset$, we denote by $x \subseteq y$ that x is a **subset** of y ,
- the **power set** of x is $\mathcal{P}(x) = \{y \mid y \subseteq x\}$,
- the **union** of x is $\bigcup x = \{z \mid \exists y(z \in y \wedge y \in x)\}$,
- a **cover** of a set x is a set $y \subseteq \mathcal{P}(x) \setminus \{\emptyset\}$ with $\bigcup y = x$. If, moreover, all sets in y are mutually disjoint, then y is a **partition** of x .

Relations

- An *ordered pair* is $(x, y) = \{x, \{x, y\}\}$, so $(x, x) = \{x, \{x\}\}$,
an *ordered n -tuple* is $(x_0, \dots, x_{n-1}) = ((x_0, \dots, x_{n-2}), x_{n-1})$ for $n > 2$,
- the *Cartesian product* of a and b is $a \times b = \{(x, y) \mid x \in a, y \in b\}$,
the *Cartesian power* of x is $x^0 = \{\emptyset\}$, $x^1 = x$, $x^n = x^{n-1} \times x$ for $n > 1$,
- the *disjoint union* of x and y is $x \uplus y = (\{\emptyset\} \times x) \cup (\{\{\emptyset\}\} \times y)$,
- a *relation* is a set R of ordered pairs, instead of $(x, y) \in R$ we usually
write $R(x, y)$ or $x R y$,

the *domain* of R is $\text{dom}(R) = \{x \mid \exists y (x, y) \in R\}$,

the *range* of R is $\text{rng}(R) = \{y \mid \exists x (x, y) \in R\}$,

the *extension* of x in R is $R[x] = \{y \mid (x, y) \in R\}$,

the *inverse relation* to R is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$,

the *restriction* of R to the set z is $R \upharpoonright z = \{(x, y) \in R \mid x \in z\}$,

- the *composition* of relations R and S is the relation

$$R \circ S = \{(x, z) \mid \exists y ((x, y) \in R \wedge (y, z) \in S)\},$$

- the *identity* on a set z is the relation $\text{Id}_z = \{(x, x) \mid x \in z\}$.

Equivalences

- A relation R on X is an *equivalence* if for every $x, y, z \in X$

$$R(x, x) \quad (\text{reflexivity})$$

$$R(x, y) \rightarrow R(y, x) \quad (\text{symmetry})$$

$$R(x, y) \wedge R(y, z) \rightarrow R(x, z) \quad (\text{transitivity})$$

- $R[x]$ is called the *equivalence class* of x in R , denoted also $[x]_R$.
- $X/R = \{R[x] \mid x \in X\}$ is the *quotient set* of X by R .
- It holds that X/R is a partition of X since the equivalence classes are mutually disjoint and cover X .
- On the other hand, a partition S of X determines the equivalence (on X)

$$\{(x, y) \mid x \in z, y \in z \text{ for some } z \in S\}.$$

Orders

Let $\leq$ be a relation on a set X . We say that $\leq$ is

- a **partial order** (of the set X) if for every $x, y, z \in X$

$$x \leq x \quad (\text{reflexivity})$$

$$x \leq y \wedge y \leq x \rightarrow x = y \quad (\text{antisymmetry})$$

$$x \leq y \wedge y \leq z \rightarrow x \leq z \quad (\text{transitivity})$$

- a **linear (total) order** if, moreover, for every $x, y \in X$

$$x \leq y \vee y \leq x \quad (\text{dichotomy})$$

- a **well-order** if, moreover, every non-empty subset of X has a **least** element.

Let us write ' $x < y$ ' for ' $x \leq y \wedge x \neq y$ '. A linear order $\leq$ on X is

- a **dense order** if X is not a singleton and for every $x, y \in X$

$$x < y \rightarrow \exists z (x < z \wedge z < y) \quad (\text{density})$$

Functions

A relation f is a **function** if every $x \in \text{dom}(f)$ has exactly one y with $(x, y) \in f$.

- We say that y is the **value** of the function f at x , denoted by $f(x) = y$,
- $f: X \rightarrow Y$ denotes that f is a function with $\text{dom}(f) = X$ and $\text{rng}(f) \subseteq Y$,
- a function f is a **surjection** (**onto** Y) if $\text{rng}(f) = Y$,
- a function f is **injection** (**one-to-one**) if for every $x, y \in \text{dom}(f)$

$$x \neq y \rightarrow f(x) \neq f(y)$$

- $f: X \rightarrow Y$ is **bijection** from X to Y if it is both injection and surjection,
- if $f: X \rightarrow Y$ is injective, then $f^{-1} = \{(y, x) \mid (x, y) \in f\}$ is its **inverse**,
- the **image** of the set A under f is $f[A] = \{y \mid (x, y) \in f \text{ for some } x \in A\}$,
- if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, their **composition** $(f \circ g): X \rightarrow Z$ satisfies

$$(f \circ g)(x) = g(f(x))$$

- XY denotes the set of all functions from X to Y .

Numbers

We give examples of standard formal constructions.

- The **natural numbers** are defined inductively by $n = \{0, \dots, n-1\}$, thus

$$0 = \emptyset, \quad 1 = \{0\} = \{\emptyset\}, \quad 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \quad \dots$$

- the set of **natural** numbers $\mathbb{N}$ is defined as the smallest set containing $\emptyset$ which is closed under $S(x) := x \cup \{x\}$ (**successor**),

- the set of **integers** is $\mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \sim$, where $\sim$ is the equivalence

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c$$

- the set of **rational** numbers is $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) / \approx$, where $\approx$ is given by

$$(a, b) \approx (c, d) \text{ if and only if } a \cdot d = b \cdot c$$

- the set of **real** numbers $\mathbb{R}$ is the set of **cuts** of rational numbers, that is non-trivial downwards closed subsets of $\mathbb{Q}$ with no **greatest** element.

($A \subset \mathbb{Q}$ is **downwards closed** if $y < x \in A$ implies $y \in A$.)

Cardinalities

- x has *cardinality smaller or equal* to the cardinality of y if there is an injective function $f: x \rightarrow y$, $(x \preceq y)$
- x has *same cardinality* as y if there is a bijection $f: x \rightarrow y$, $(x \approx y)$
- x has *cardinality strictly smaller* than y if $x \preceq y$ but not $x \approx y$, $(x \prec y)$

Theorem (Cantor) $x \prec \mathcal{P}(x)$ for every set x .

Proof $f(y) = \{y\}$ for $y \in x$ is an injective function $f: x \rightarrow \mathcal{P}(x)$, so $x \preceq \mathcal{P}(x)$. Suppose for a contradiction that there is an injective $g: \mathcal{P}(x) \rightarrow x$. Define

$$y = \{g(z) \mid z \subseteq x \wedge g(z) \notin z\}$$

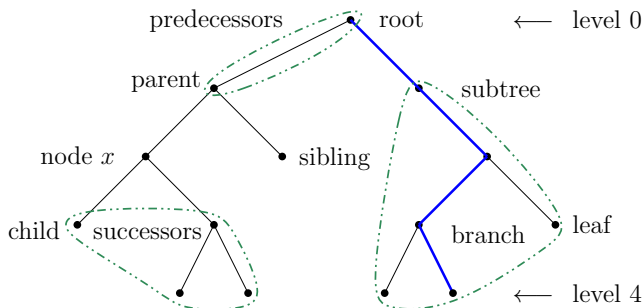
By definition, $g(y) \in y$ if and only if $g(y) \notin y$, a contradiction. $\square$

- for every x there is *cardinal number* κ with $x \approx \kappa$, denoted by $|x| = \kappa$,
- x is *finite* if $|x| = n$ for some $n \in \mathbb{N}$; otherwise, x is *infinite*,
- x is *countable* if x is finite or $|x| = |\mathbb{N}| = \omega$; otherwise, x is *uncountable*,
- x has *cardinality of the continuum* if $|x| = |\mathcal{P}(\mathbb{N})| = \mathfrak{c}$.

n -ary relations and functions

- A relation of **arity** $n \in \mathbb{N}$ on X is any set $R \subseteq X^n$, so for $n = 0$ we have either $R = \emptyset = 0$ or $R = \{\emptyset\} = 1$, and for $n = 1$ we have $R \subseteq X$,
- A (partial) function of **arity** $n \in \mathbb{N}$ from X to Y is any function $f \subseteq X^n \times Y$. We say that f is **total** on X^n if $\text{dom}(f) = X^n$, denoted by $f: X^n \rightarrow Y$. If, moreover, $Y = X$, we say that f is an **operation** on X .
- A function $f: X^n \rightarrow Y$ is **constant** if $\text{rng}(f) = \{y\}$ for some $y \in Y$, for $n = 0$ we have $f = \{(\emptyset, y)\}$ and we identify f with the **constant** y .
- The arity of a relation or function is denoted by $\text{ar}(R)$ or $\text{ar}(f)$ and we speak about **nullary**, **unary**, **binary**, etc. relations and functions.

Trees



- A **tree** is a set T with a partial order $<_T$ in which there is a unique least element, called the **root**, and the set of predecessors of any element is **well ordered** by $<_T$,
- a **branch** of a tree T is a **maximal** linearly ordered subset of T ,
- we adopt standard terminology on trees from the graph theory, e.g.

a branch in a finite tree is a path from the root to a leaf.

König's lemma

We will consider (*for simplicity*) usually finitely branching trees in which every node except the root has an **immediate** predecessor (*parent*).

- *n -th level* of a tree T for $n \in \mathbb{N}$ is given by induction, it is the set of children of nodes from the $(n - 1)$ -th level, 0-th level containing exactly the root,
- the *depth* of T is the maximal $n \in \mathbb{N}$ of non-empty level;
if T has infinite branch, then it has *infinite depth* ω .
- a tree T is *n -ary* for $n \in \mathbb{N}$ if every node has **at most** n children.
It is *finitely branching* if every node has only finitely many children.

Lemma (König) *Every infinite, finitely branching tree contains an infinite branch.*

Proof We start in the root. Since it has only finitely many children, there exists a child with infinitely many successors. We *choose* it and continue in its subtree. In this way we construct an infinite branch. □

Ordered trees

- An *ordered tree* is a tree T with a linear order of children at each node. These orders are called *left-right orders* and are denoted by $<_L$. In comparison with $<_L$, the order $<_T$ is called the *tree order*.
- A *labeled tree* is a tree T with an arbitrary function (a *labeling function*), that assigns to each node some object (a *label*).
- Labeled ordered trees represent, for example, structure of formulas.

