

Propositional and Predicate Logic - Tutorial 1,(2)

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1. Consider a finite game of two alternating players. Assume the game ends after n rounds by win of one of the players, denoted by X, Y where X begins. The game is given by formula $\varphi(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ expressing that the game with moves $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ ends by a win of X (there are no draws). Find formulas (with use of quantifiers) that expresses
 - (a) “ X cannot lose”, “ Y cannot lose”,
 - (b) “ X has a winning strategy”, “ Y has a winning strategy”.
2. Assume we are given an (undirected) graph G and two vertices u, v . Find a propositional formula that is satisfiable if and only if
 - (a) G is bipartite,
 - (b) G is 3-colorable,
 - (c) G has a perfect matching,
 - (d) there is a path between u and v in G .
3. Find first-order formulas (in the language of graph theory) expressing about an (undirected) graph that
 - (a) “ u and v have (a / exactly one / at most one) common neighbor”,
 - (b) “there are at least three mutually independent edges”,
 - (c) “there is a path of length n between u and v , where given $n > 0$ is fixed.
4. Find second-order formulas (in the language of graph theory) expressing about an (undirected) graph that
 - (a) “there exists a bipartition”,
 - (b) “there exists a perfect matching”,
 - (c) “there exists a path between u and v ”.
5. Find first-order formulas (with the symbol $\leq$) expressing about a partially ordered set
 - (a) “ x is the smallest element”, “ x is a minimal element”,
 - (b) “ x has an immediate successor”,
 - (c) “every two elements have the greatest common predecessor”.
6. Find first order formulas (with use of equality) expressing for a fixed $n > 0$ that
 - (a) “there exist at least n elements”,
 - (b) “there exist at most n elements”,
 - (c) “there exist exactly n elements”

Is it possible to express with use of (possibly infinite) set of formulas that “there are infinitely many elements”?

7. Find a second-order formula expressing “there exist finitely many elements”. Hint:
 - (a) Find first-order formulas (with a symbol f for a function) expressing “ f is injective”, “ f is surjective”.
 - (b) Find a second-order formula expressing “every function that is surjective is also injective”.

8. Can we color the integers from 1 to n with two colors such that there is no monochromatic solution of an equation $a + b = c$ with $1 \leq a < b < c \leq n$? Write a proposition φ_n for $n = 8$ that is satisfiable if and only if such coloring exists.
9. (*Pigeonhole principle*). Let $n \geq 2$ be a fixed natural number. Assume that we have n pigeons and $n - 1$ pigeonholes. We want to express that
 - (i) Every pigeon sits in some pigeonhole,
 - (ii) there is no pigeonhole with more than one pigeon sitting in it.

Let $\mathbb{P} = \{p_j^i \mid 1 \leq i \leq n, 1 \leq j \leq n - 1\}$ be a set of propositional variables, where p_j^i represents that “the i -th pigeon sits in the j -th pigeonhole”.

- (a) Write propositions φ_i and ψ_j over $\mathbb{P}$ expressing that “the i -th pigeon sits in some pigeonhole” and “in the j -th pigeonhole sits not more than one pigeon”, respectively, where $1 \leq i \leq n, 1 \leq j \leq n - 1$. Write a set of propositions expressing (i) and (ii).
10. Three proposals are being discussed in the parliament: *school charges*, *tax increase*, *restriction of smoking in restaurants*.
 - (i) The party A demands that in case the party B or the party C has his demand fulfilled, there will be no school charges or no tax increase.
 - (ii) The party B wants to restrict smoking if the party C does not have his demand fulfilled or tax do not increase.
 - (iii) The party C requires that in case the party A has his demand fulfilled, there will be no tax increase and no smoking restriction.
 - (iv) In the final voting exactly two parties had their demands fulfilled.

Let the propositional letters p, q, r represent (respectively) that the proposals on *school charges*, *tax increase*, *smoking restriction* have been passed. Furthermore, let a, b, c represent (respectively) that each party’s demand has been fulfilled and let $\mathbb{P} = \{p, q, r, a, b, c\}$.

- (a) Write propositions $\varphi_1, \varphi_2, \varphi_3$ in the form of an equivalence and a proposition φ_4 over $\mathbb{P}$ that express (respectively) (i), (ii), (iii), and (iv).
11. (In Shakespeare’s play *The Merchant of Venice*) those who wish to win a beautiful Portia’s hand in marriage need to find out which of the three caskets, made of gold, silver, and lead, hides Portia’s portrait. We know that
 - (i) Portia’s portrait is in exactly one casket.
 - (ii) At most one of the inscriptions on the caskets is true.
 - (iii) The inscription on the golden casket says: “*The portrait is not in this casket.*”
 - (iv) The inscription on the silver casket says: “*If the inscription on the golden casket is true, then the portrait is in the leaden casket.*”
 - (v) The inscription on the leaden casket says: “*The inscription on the golden casket is false.*”

Let the propositional letters g, s, l represent (respectively) that “the portrait is in golden / silver / leaden casket” and letters t_g, t_s, t_l represent (respectively) that “the inscription on golden / silver / leaden casket is true.” Furthermore, let $\mathbb{P}' = \{g, s, l, t_g, t_s, t_l\}$.

- (a) Write propositions φ_1, φ_2 over $\mathbb{P}'$ expressing the statements (i), (ii), and propositions (in form of equivalences) $\varphi_3, \varphi_4, \varphi_5$ over $\mathbb{P}'$ representing (respectively) our knowledge from (iii), (iv), (v).