

Predicate and Propositional Logic - Tutorials 2,3

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1. Consider a theory $T = \{\neg q \rightarrow (\neg p \vee q), \neg p \rightarrow q, r \rightarrow q\}$. Which of the following propositions are valid, contradictory, independent, satisfiable, equivalent in T ?

- (a) p, q, r, s
- (b) $p \vee q, p \vee r, p \vee s, q \vee s$
- (c) $p \wedge q, q \wedge s, p \rightarrow q, s \rightarrow q$

2. Prove or disprove that the following sets of connectives are universal.

- (a) $\{\downarrow\}$ where $\downarrow$ is Peirce arrow (NOR)
- (b) $\{\uparrow\}$ where $\uparrow$ is Sheffer stroke (NAND)
- (c) $\{\vee, \rightarrow, \leftrightarrow\}, \{\vee, \wedge, \rightarrow\}$

3. Transform the following propositions into DNF and CNF a) by using truth tables (determining the models), b) by using transformation rules.

- (a) $(\neg p \vee q) \rightarrow (\neg q \wedge r)$
- (b) $(\neg p \rightarrow (\neg q \rightarrow r)) \rightarrow p$
- (c) $((p \rightarrow \neg q) \rightarrow \neg r) \rightarrow \neg p$

4. Let $\text{maj}_n: (\{0, 1\}^n)^3 \rightarrow \{0, 1\}^n$ be the coordinate-wise majority function; that is, for example

$$\text{maj}_4((0, 1, 0, 1), (1, 1, 0, 0), (1, 1, 0, 0)) = (1, 1, 0, 0)$$

We say that a set $K \subseteq \{0, 1\}^n$ is a *median* set if it is closed under maj_n .

- (a) Show that for every 2-CNF proposition φ it holds that $M(\varphi)$ is a median set.
 - (b)* Show that for every median set $K \subseteq \{0, 1\}^n$ there exists a 2-CNF proposition φ over n variables such that $M(\varphi) = K$.
5. Consider an infinite theory $T = \{p_i \rightarrow (p_{i+1} \vee q_{i+1}), q_i \rightarrow (p_{i+1} \vee q_{i+1}) \mid i \in \mathbb{N}\}$ over $\text{var}(T)$.
 - (a) Which propositions in the form $p_i \rightarrow p_j$ are logical consequences of T ?
 - (b) Which propositions in the form $p_i \rightarrow (p_j \vee q_j)$ are logical consequences of T ?
 - (c) Determine all models of the theory T .
 6. Prove or disprove (or find the correct relation) that for every theory T and propositions φ, ψ over $\mathbb{P}$ it holds
 - (a) $T \models \varphi$, if and only if $T \not\models \neg\varphi$
 - (b) $T \models \varphi$ and $T \models \psi$, if and only if $T \models \varphi \wedge \psi$
 - (c) $T \models \varphi$ or $T \models \psi$, if and only if $T \models \varphi \vee \psi$
 - (d) $T \models \varphi \rightarrow \psi$ and $T \models \psi \rightarrow \chi$, if and only if $T \models \varphi \rightarrow \chi$
 7. Prove or disprove (or find the correct relation). For every theories T and S over $\mathbb{P}$
 - (a) $S \subseteq T \Rightarrow \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(S)$
 - (b) $\theta^{\mathbb{P}}(S \cup T) = \theta^{\mathbb{P}}(S) \cup \theta^{\mathbb{P}}(T)$
 - (c) $\theta^{\mathbb{P}}(S \cap T) = \theta^{\mathbb{P}}(S) \cap \theta^{\mathbb{P}}(T)$
 8. Let $|\mathbb{P}| = n$ and $\varphi \in \text{VF}_{\mathbb{P}}$ with $|M(\varphi)| = m$.

- (a) What is the number of nonequivalent propositions ψ such that $\varphi \models \psi$ or $\psi \models \varphi$?
 - (b) What is the number of nonequivalent theories over $\mathbb{P}$ in which φ is valid? What is the number of nonequivalent *complete* theories over $\mathbb{P}$ in which φ is valid?
 - (c) What is the number of nonequivalent theories T over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is satisfiable?
 - (d) Let, moreover, $\{\varphi, \psi\}$ be an unsatisfiable theory with $|M(\psi)| = p$. What is the number of nonequivalent propositions χ such that $\varphi \vee \psi \models \chi$? What is the number of nonequivalent theories in which $\varphi \vee \psi$ is valid?
9. Let $T = \{q \rightarrow (\neg p \rightarrow r), \neg r \rightarrow (\neg p \wedge q), (s \rightarrow r) \rightarrow p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
- (a) Axiomatize the theory T by a proposition in CNF.
 - (b) Find all models of the theory T .
 - (c) Is the theory T an extension of the theory $S = \{q \leftrightarrow \neg r\}$ over the language $\{q, r\}$? Is T a conservative extension of S ? Justify.
 - (d) Determine the number of mutually inequivalent propositions in the language $\mathbb{P}$ that are *contradictory* in both the theories T and S . Justify.
10. Let $T = \{p \vee q \rightarrow r, \neg(p \rightarrow \neg s)\}$ be a theory over the propositional language $\mathbb{P} = \{p, q, r, s\}$.
- (a) Is the proposition $q \rightarrow p$ valid in the theory T ? Is it contradictory? Is it independent? Justify.
 - (b) Find all models of the theory T .
 - (c) Find a theory S over the language $\mathbb{P}' = \{p, q, r\}$ such that T is a conservative extension of S . Axiomatize S by a proposition in CNF. Justify why S has the desired property.
 - (d) Determine the number of mutually inequivalent propositions φ over $\mathbb{P}$ such that φ is valid in T and independent in S .
11. Let $T = \{p, \neg q \rightarrow \neg r, \neg q \rightarrow \neg s, r \rightarrow p, \neg s \rightarrow \neg p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
- (a) Using the implication graph, show that T is satisfiable.
 - (b) Find all models of the theory T and axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.
 - (c) Determine, and justify, the number of mutually
 - (i) inequivalent propositions over $\mathbb{P}$ that are independent in T ,
 - (ii) T -inequivalent propositions over $\mathbb{P}$ which are independent in T .
12. Let $T = \{(r \rightarrow p) \rightarrow \neg q, \neg q \rightarrow p, \neg(r \wedge q), r \rightarrow \neg s\}$ be a theory in the language $\mathbb{P} = \{p, q, r, s\}$.
- (a) Axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.
 - (b) Is the theory T a conservative extension of some theory over the language $\{p, q, r\}$? Justify.
 - (c) Determine the number of mutually inequivalent noncontradictory extensions of the theory T over the language $\{p, q, r, s, t\}$. Justify.
13. Let $T = \{p \rightarrow \neg q \wedge r, q \vee r, (q \wedge s) \leftrightarrow r\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
- (a) Is the proposition $q \rightarrow p$ valid in the theory T ? Is it contradictory? Is it independent? Justify your answers.
 - (b) Axiomatize $M(T)$ by a proposition in CNF.
 - (c) Determine the number of mutually inequivalent theories S over $\mathbb{P}' = \{r, s\}$ such that T is a conservative extension of S . How many of them are complete?