Predicate and Propositional Logic - Tutorial 5

Oct 29, 2025

Examples for the midterm test - Homework.

1. In the presidental elections we have two candidates, Mr. A and Mr. B.

   (*i*) *Mr. A says: "I will be elected or Mr. B lies."*

   (*ii*) *Mr. B says: "Mr. A will not be elected or I lie."*

   (*iii*) *Exactly one candidate will be elected.*

   Let the propositional atoms e_A, e_B represent that *Mr. A (resp. Mr. B) will be elected* and let t_A, t_B represent that *Mr. A (resp. Mr. B) speaks the truth.* Let us denote $\mathbb{P} = \{e_A, e_B, t_A, t_B\}$.

   (a) Write propositions φ_1, φ_2 in the form of equivalence and a proposition φ_3 in CNF expressing (in this order) (*i*), (*ii*), (*iii*), all over the language $\mathbb{P}$. *(20p)*

   (b) Let $T = \{\varphi_1, \varphi_2, \varphi_3\}$. Prove by tableau method that $T \models e_B$. *(40p).*

   (c) Give an example of a proposition over $\mathbb{P}$ that is independent in theory T, or show that such proposition does not exist. *(20p).*

   (d) Find a theory S over $\{e_A, e_B\}$ such that T is a conservative extension of S. *(20p)*

2. *(Pigeonhole principle).* Let $n \geq 2$ be a fixed natural number. Assume that we have n pigeons and $n-1$ pigeonholes. We want to show (by resolution) that the following two statements cannot both be true:

   (*i*) *Every pigeon sits in some pigeonhole,*

   (*ii*) *there is no pigeonhole with more than one pigeon sitting in it.*

   Let $\mathbb{P} = \{p^i_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ be a set of propositional variables, where p^i_j represents that *"the i-th pigeon sits in the j-th pigeonhole"*.

   (a) Write propositions φ_i and ψ_j over $\mathbb{P}$ expressing that *"the i-th pigeon sits in some pigeonhole"* and *"in the j-th pigeonhole sits not more than one pigeon"*, respectively, where $1 \leq i \leq n$, $1 \leq j \leq n-1$. Using the propositions φ_i and ψ_j construct a theory T_n expressing (*i*) and (*ii*). *(20p)*

   (b) Now let $n = 3$ and $T' = T_3 \cup \{p^1_1\}$, that is, we additionally assume that *"the 1st pigeon sits in the 1st pigeonhole"*. Convert T' to CNF. *(20p)*

   (c) Show by tableux method (or by resolution) that T' is unsatisfiable. *(40p)*

   (d) Let $T^* = T' \setminus \{\psi_2\}$ be a theory over $\mathbb{P}$. Is the theory T' a conservative extension of the theory T^*? Justify your answer. *(20p)*

3. Let the propositional variables w, m, r represent (respectively) that *"people wear umbrellas"*, *"meteorologists forecast rain"*, *"it rains"* and denote $\mathbb{P} = \{w, m, r\}$. We know that

   (*i*) *If it does not rain, people do not wear umbrellas.*

   (*ii*) *If meteorologists forecast rain, people wear umbrellas.*

   (*iii*) *If meteorologists do not forecast rain, it rains.*

   (a) Write propositions φ_1, φ_2, φ_3 in $\mathbb{P}$ expressing (respectively) (*i*), (*ii*), (*iii*), transform them into 2-CNF and use the implication graph to show that the theory $T = \{\varphi_1, \varphi_2, \varphi_3\}$ is not contradictory. *(30 points)*

   (b) Show by tableux method (or resolution) that from T it follows *it rains. (30 points)*

(c) Is the theory T an extension of the theory S over $\{w, m\}$ with the following single axiom (iv)? Is T a conservative extension of S? Justify your answers. *(20 points)*

(iv) *People wear umbrellas only if meteorologists forecast rain.*

(d) How many there exist mutually non-equivalent propositions over $\mathbb{P}$ that are independent in T? Write at least two of them, if they exist. *(20 points)*

4. (In Shakespeare's play The Merchant of Venice) those who wish to win a beautiful Portia's hand in marriage need to find out (by resolution) which of the three caskets, made of gold, silver, and lead, hides Portia's portrait. We know that

(i) Portia's portrait is in exactly one casket.

(ii) At most one of the inscriptions on the caskets is true.

(iii) The inscription on the golden casket says: *"The portrait is not in this casket."*

(iv) The inscription on the silver casket says: *"If the inscription on the golden casket is true, then the portrait is in the leaden casket."*

(v) The inscription on the leaden casket says: *"The inscription on the golden casket is false."*

Let the propositional letters g, s, l represent (respectively) that *"the portrait is in golden / silver / leaden casket"* and letters t_g, t_s, t_l represent (respectively) that *"the inscription on golden / silver / leaden casket is true."* Furthermore, let $\mathbb{P}' = \{g, s, l, t_g, t_s, t_l\}$.

(a) Write propositions φ_1, φ_2 over $\mathbb{P}'$ expressing the statements (i), (ii), and propositions (in form of equivalences) φ_3, φ_4, φ_5 over $\mathbb{P}'$ representing (respectively) our knowledge from (iii), (iv), (v). *(20p)*

(b) Applying T'-equivalent transformations (replacing a subformula with a T'-equivalent formula) find a theory T over $\mathbb{P} = \{g, s, l\}$ such that the theory $T' = \{\varphi_1, \ldots, \varphi_5\}$ over $\mathbb{P}'$ is a conservative extension of T. *(20p).*

(c) Find out by tableux method from T which casket contains Porcia's portrait. *(40p)*

(d) What is the number of (mutually nonequivalent) extensions of the theory T' over $\mathbb{P}'$? How many of them are complete? Give an explanation. *(20p)*

5. Let $T = \{p \to \neg q \wedge r, q \vee r, (q \wedge s) \leftrightarrow r\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

(a) Construct a finished tableau from the theory T with the entry $T(q \to p)$ in the root. *(40p)*

(b) Is the proposition $q \to p$ valid in the theory T? Is it contradictory? Is it independent? Justify your answers (to all three questions). *(20p)*

(c) Axiomatize $M(T)$ by a proposition in CNF. *(20p)*

(d) Determine the number of mutually inequivalent theories S over $\mathbb{P}' = \{r, s\}$ such that T is a conservative extension of S. How many of them are complete? Justify your answer. *(20p)*