

Predicate and Propositional Logic - Tutorial 7

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1. Which of the variable occurrences are free/bound in the following formulas? Find variants of these formulas without variables that have both free and bound occurrence.
 - (a) $(\exists x)(\forall y)P(y, z) \vee (y = 0)$
 - (b) $(\exists x)(P(x) \wedge (\forall x)Q(x)) \vee (x = 0)$
 - (c) $(\exists x)(x > y) \wedge (\exists y)(y > x)$
2. Let φ denote the formula $(\forall x)((x = z) \vee (\exists y)(f(x) = y) \vee (\forall z)(y = f(z)))$. Which of the following terms are substitutable into φ ?
 - (a) the term z for the variable x , the term y for the variable x ,
 - (b) the term z for the variable y , the term $2 * y$ for the variable y ,
 - (c) the term x for the variable z , the term y for the variable z ,
3. Are the following formulas variants of the formula $(\forall x)(x < y \vee (\exists z)(z = y \wedge z \neq x))$?
 - (a) $(\forall z)(z < y \vee (\exists z)(z = y \wedge z \neq z))$
 - (b) $(\forall y)(y < y \vee (\exists z)(z = y \wedge z \neq y))$
 - (c) $(\forall u)(u < y \vee (\exists z)(z = y \wedge z \neq u))$
4. Let $\mathcal{A} = (\{a, b, c, d\}, \triangleright^A)$ be a structure for the language with a single binary relation symbol $\triangleright$, where $\triangleright^A = \{(a, c), (b, c), (c, c), (c, d)\}$. Which of the following formulas are valid in $\mathcal{A}$?
 - (a) $x \triangleright y$
 - (b) $(\exists x)(\forall y)(y \triangleright x)$
 - (c) $(\exists x)(\forall y)((y \triangleright x) \rightarrow (x \triangleright x))$
 - (d) $(\forall x)(\forall y)(\exists z)((x \triangleright z) \wedge (z \triangleright y))$
 - (e) $(\forall x)(\exists y)((x \triangleright z) \vee (z \triangleright y))$
5. For every formula φ from the previous exercise find a structure $\mathcal{B}$ (if it exists) such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \not\models \varphi$.
6. Are the following sentences valid / contradictory / independent (in logic)?
 - (a) $(\exists x)(\forall y)(P(x) \vee \neg P(y))$
 - (b) $(\forall x)(P(x) \rightarrow Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$
 - (c) $(\forall x)(P(x) \vee Q(x)) \rightarrow ((\forall x)(P(x) \vee (\forall x)Q(x)))$
 - (d) $(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x)P(x) \rightarrow (\exists x)Q(x))$
 - (e) $(\exists x)(\forall y)P(x, y) \rightarrow (\forall y)(\exists x)P(x, y)$
7. Prove (semantically) the following claims. For every structure $\mathcal{A}$, formula φ , and sentence ψ ,
 - (a) $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$
 - (b) $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \rightarrow \varphi)$
 - (c) $\mathcal{A} \models ((\exists x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \rightarrow \psi)$
 - (d) $\mathcal{A} \models ((\forall x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \rightarrow \psi)$

Does this hold also for every formula ψ with a free variable x ? And for every formula ψ in which x is not free?

8. Determine whether the following holds for every formula φ .

- (a) $\varphi \models (\forall x)\varphi$
- (b) $\models \varphi \rightarrow (\forall x)\varphi$
- (c) $\varphi \models (\exists x)\varphi$
- (d) $\models \varphi \rightarrow (\exists x)\varphi$

9. The theory of groups T is of language $L = \langle +, -, 0 \rangle$ with equality where $+$ is a binary function symbol, $-$ is a unary function symbol, 0 is a constant symbol, and has it axioms

$$\begin{aligned}x + (y + z) &= (x + y) + z \\0 + x &= x = x + 0 \\x + (-x) &= 0 = (-x) + x\end{aligned}$$

Are the following formulas valid / contradictory / independent in T ?

- (a) $x + y = y + x$
 - (b) $x + y = x \rightarrow y = 0$
 - (c) $x + y = 0 \rightarrow y = -x$
 - (d) $-(x + y) = (-y) + (-x)$
10. Consider a structure $\underline{\mathbb{Z}}_4 = \langle \{0, 1, 2, 3\}, +, -, 0 \rangle$ where $+$ is the binary addition modulo 4 and $-$ is the unary function for the *inverse* element of $+$ with respect to the *neutral* element 0.
- (a) Is $\underline{\mathbb{Z}}_4$ a model of the theory T from the previous example (i.e. is it a *group*)?
 - (b) Determine all substructures $\underline{\mathbb{Z}}_4 \langle a \rangle$ generated by some $a \in \underline{\mathbb{Z}}_4$.
 - (c) Does $\underline{\mathbb{Z}}_4$ contain also other substructures?
 - (d) Is every substructure of $\underline{\mathbb{Z}}_4$ a model of T ?
 - (e) Is every substructure of $\underline{\mathbb{Z}}_4$ elementary equivalent to $\underline{\mathbb{Z}}_4$?
 - (f) Is every substructure of a *commutative* group (i.e. a group that satisfies also 9(a)) again a commutative group?
11. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ be the structure of rational numbers with standard operations (thus forming a *field*).
- (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T from the previous exercises?
 - (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ to a model of T ?
 - (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?
 - (d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences that are valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?
12. Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.
- (a) Is T (semantically) consistent?
 - (b) Are all models of T elementary equivalent? That is, is T (semantically) complete?
 - (c) Find all simple complete extensions of T .
 - (d) Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T ? Is T' a simple extension of T ? Is T' a conservative extension of T ?