

Predicate and Propositional Logic - Tutorial 9

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- Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ be the structure of rational numbers with standard operations (thus forming a *field*) and let T be the theory of groups.
 - Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T ?
 - Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ to a model of T ?
 - Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?
 - Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?
- Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.
 - Is T (semantically) consistent?
 - Are all models of T elementarily equivalent? That is, is T (semantically) complete?
 - Find all simple complete extensions of T .
 - Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T ? Is T' a simple extension of T ? Is T' a conservative extension of T ?
- Let T' be the extension of $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ in $L = \langle +, 0, \leq \rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$\begin{aligned} -x = y &\leftrightarrow x + y = 0 \\ x < y &\leftrightarrow x \leq y \wedge \neg(x = y) \end{aligned}$$

Find formulas of L that are equivalent in T' to the following formulas.

- $x + (-x) = 0$
 - $x + (-y) < x$
 - $-(x + y) < -x$
- Consider the following database as a relational structure $\mathcal{D} = \langle D, Movies, Program, c^D \rangle_{c \in D}$ of language $L = \langle F, P, c \rangle_{c \in D}$ with equality where $D = \{\text{'Po strništi bos'}, \text{'J. Trška'}, \text{'Mat'}, \text{'13:15'}, \dots\}$ and $c^D = c$ for every $c \in D$. Write formulas that define in $\mathcal{D}$ tables of
 - movies in which a director is acting,
 - cinemas and times where and when one can see a movie in which a director is acting,
 - directors that act in movies that are on program in the cinema Mat,
 - actors or directors whose movie is not on a program in any cinema.

Movie	name	director	actor	Program	cinema	name	time
	Lidé z Maringotek	M. Frič	J. Trška		Světazor	Po strništi bos	13:15
	Po strništi bos	J. Svěrák	Z. Svěrák		Mat	Po strništi bos	16:15
	Po strništi bos	J. Svěrák	J. Trška		Mat	Lidé z Maringotek	18:30
	...	...	...		...	...	...

- Let $L = \langle F \rangle$ be a language with equality where F is a binary function symbol. Write formulas that define (without parameters) the following sets in the following structures:
 - the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$ where $\cdot$ is the standard multiplication of real numbers,
 - the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure $\mathcal{A}$,
 - the set of all at most one-element subsets of $\mathbb{N}$ in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$.

6. Assume that

- (a) all guilty persons are liars,
- (b) at least one of the accused is also a witness,
- (c) no witness lies.

Prove by tableau method that not all accused are guilty.

7. Let $L(x, y)$ represent that “there is a flight from x to y ” and let $S(x, y)$ represent that “there is a connection from x to y ”. Assume that

- (a) From Prague you can fly to Bratislava, London and New York, and from New York to Paris,
- (b) $(\forall x)(\forall y)(L(x, y) \rightarrow L(y, x))$,
- (c) $(\forall x)(\forall y)(L(x, y) \rightarrow S(x, y))$,
- (d) $(\forall x)(\forall y)(\forall z)(S(x, y) \wedge L(y, z) \rightarrow S(x, z))$.

Prove by tableau method that there is a connection from Bratislava to Paris.

8. Let φ, ψ be sentences or formulas in a free variable x , denoted by $\varphi(x), \psi(x)$. Find tableau proofs of the following formulas.

- (a) $(\exists x)(\varphi(x) \vee \psi(x)) \leftrightarrow (\exists x)\varphi(x) \vee (\exists x)\psi(x)$,
- (b) $(\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$,
- (c) $(\varphi \vee (\forall x)\psi(x)) \rightarrow (\forall x)(\varphi \vee \psi(x))$ where x is not free in φ ,
- (d) $(\varphi \wedge (\exists x)\psi(x)) \rightarrow (\exists x)(\varphi \wedge \psi(x))$ where x is not free in φ .
- (e) $(\exists x)(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow (\exists x)\psi(x))$ where x is not free in φ ,
- (f) $(\exists x)(\varphi \wedge \psi(x)) \rightarrow (\varphi \wedge (\exists x)\psi(x))$ where x is not free in φ ,
- (g) $(\exists x)(\varphi(x) \rightarrow \psi) \rightarrow ((\forall x)\varphi(x) \rightarrow \psi)$ where x is not free in ψ ,
- (h) $((\exists x)\varphi(x) \rightarrow \psi) \rightarrow (\forall x)(\varphi(x) \rightarrow \psi)$ where x is not free in ψ .

9. Let T^* be a theory with axioms of equality. Prove by tableau method that

- (a) $T^* \models x = y \rightarrow y = x$ (symmetry of $=$)
- (b) $T^* \models (x = y \wedge y = z) \rightarrow x = z$ (transitivity of $=$)

Hint: To show (a) apply the axiom of equality (iii) for $x_1 = x, x_2 = x, y_1 = y$ a $y_2 = x$, to show (b) apply (iii) for $x_1 = x, x_2 = y, y_1 = x$ a $y_2 = z$.

10. Let L be a language with equality containing a binary relation symbol $\leq$ and let T be a theory of L such that T has an infinite model and the axioms of linear ordering are valid in T . Applying the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite decreasing chain*; that is, there are elements c_i for every $i \in \mathbb{N}$ in A such that

$$\cdots < c_{n+1} < c_n < \cdots < c_0.$$

(This shows that the notion of *well-ordering* is not definable in a first-order language.)